BRNO UNIVERSITY OF TECHNOLOGY

FACULTY OF CIVIL ENGINEERING

MATHEMATICS

Mathematics Used in Specialised Courses Taught at FCE

FRV 808/2009/F1/a Project
BUT Faculty of Civil Engineering

COMPLEMENTARY STUDY MATERIAL
FOR FCE STUDENTS

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Contents

Introduction

The transition to a new Bachelor's degree study structure has brought many changes. New Bachelor's and Master's degree programmes have been accredited. Although the previous mathematical content has been preserved, its distribution across the courses has changed. While the probability and mathematical statistics courses have been shifted to the third year, numerical mathematics is now taught in the first year of the follow-up Master's degree programme .

The introduction of Bachelor's degree programmes has brought about problems with the difference between what is taught and what is required of the students as compared with the previous engineering study. This also relates to the way mathematical theory is used to justify the results of the parts taught in specialisations for which there is less time and which are viewed as too difficult by the Bachelor's students. Note that Bachelor's degree programmes take four years with most of the students continuing in the follow-up three-semester master's degree programmes. Graduates from a follow-up Master's programme should be able to read literature specific for their field that does not avoid mathematics.

Therefore, it is certainly good to remind Bachelor's students of problems encountered in specialized courses that use the mathematical background taught in the mathematics courses. This may encourage at least some of them to devote more effort to the mathematics taught in the basic courses convincing them of its importance.

This is also the main purpose of this collection of problems. What is presented here is only its first version. It should be further extended to contain more mathematical topics applied in more specialised courses taught at the Faculty of Civil Engineering.

It is our pleasure to thank all the colleagues from other faculty departments who have contributed to work on this collection.

The authors
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Chapter 1

Elementary problems

Problem 1 ([7], s. 9) *How will the length change of a bar* 3.2 *m long hanging by its upper end*? (The modulus of elasticity in tension of steel is $E = 2 \cdot 10^{11}\, Pa$, density $\rho = 7800\, kg \cdot m^{-3}$, $\Delta l = \int_0^l \frac{\rho g}{E} \cdot (l - y)\, dy$.)

Solution:

$$\Delta l = \int_0^l \frac{\rho g}{E} \cdot (l - y)\, dy = \frac{\rho g}{E} \left[ly - \frac{y^2}{2} \right]_0^l = \frac{\rho g}{E} \cdot \frac{l^2}{2} =$$

$$= \frac{7,8 \cdot 10^3\, kg \cdot\, m^{-3} \cdot 9,81\, m \cdot s^{-2} \cdot (3,2\, m)^2}{2 \cdot 10^{11}\, Pa \cdot 2} = 1,96 \cdot 10^{-6}\, m$$

Problem 2 ([7], s. 36) *A wooden cylinder is submerged in water until two thirds of its height. What work is done if the cylinder is completely pulled out of water? The cylinder radius is* $10\, cm$, *its height is* $60\, cm$, *the specific density of wood is* $\rho_D = 600\, kg \cdot m^{-3}$. *The water level in the vessel is assumed to be constant.*

Solution: The total work done when two thirds of the height are pulled out is given by

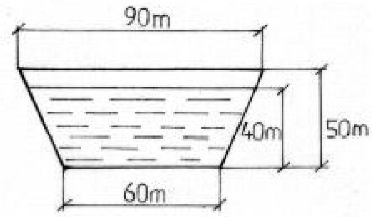
$$W = \int_0^{\frac{2}{3}h} (\pi \cdot r^2 \cdot h \cdot \rho_D \cdot g - \pi \cdot r^2 \cdot y \cdot \rho_V \cdot g)\, dy = \pi r^2 g \int_0^{\frac{2}{3}h} (h \cdot \rho_D - y \cdot \rho_V)\, dy.$$

Then

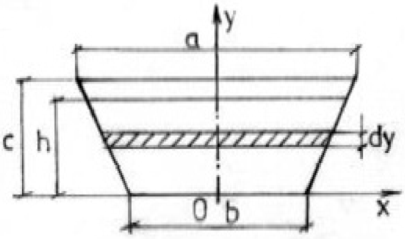
$$W = \pi r^2 g \left[h \rho_D y - \frac{y^2}{2} \rho_V \right]_0^{\frac{2}{3}h} = \pi r^2 g \left(\frac{2}{3} h^2 \rho_D - \frac{4}{18} h^2 \rho_V \right) = \frac{2}{3} \pi r^2 g h^2 (\rho_D - \frac{1}{3} \rho_V)$$

$$= \frac{2}{3} \pi (0,1\, m)^2 9,81\, m \cdot s^{-2} (0,6\, m)^2 (600\, kg \cdot m^{-3} - \frac{1}{3} 1000\, kg \cdot m^{-3}) = 19,7\, J.$$

Problem 3 ([7], p. 38, modified) *By what strength does the water press on the dam of a reservoir having the shape of a trapezium (see the below figures)?* $(F = \int_0^h \left(\frac{a-b}{2c}y + \frac{b}{2}\right) y\rho g \, dy.)$



The reservoir dam.



The dam dimensions – notation.

Solution:

$$F = \int_0^h \left(\frac{a-b}{2c}y + \frac{b}{2}\right) y\rho g \, dy =$$

$$= \rho g \frac{a-b}{2c}\left[\frac{y^3}{3}\right]_0^h + \rho g \frac{b}{2}\left[\frac{y^2}{2}\right]_0^h = \frac{1}{2}\rho g h^2 \left[\frac{a-b}{3c}h + \frac{b}{2}\right] =$$

$$= \frac{1}{2} 1000\, kg{\cdot}m^{-3}{\cdot}9,81\, m{\cdot}s^{-2}(40\, m)^2 \left[\frac{90\, m - 60\, m}{3 \cdot 50\, m} \cdot 40\, m + \frac{60\, m}{2}\right] = 2,98{\cdot}10^8\, N.$$

Problem 4 ([8], p. 8, modified) *Calculate the velocity and acceleration of the movement of a mass point described by*

$$s(t) = (0,06\, m) \cdot \cos\left((1,5\pi s^{-1})t + \frac{2}{3}\pi\right).$$

Solution: The velocity is calculated by differentiating the function of movement by time

$$v(t) = \frac{ds(t)}{dt} = s'(t) = -(0,06\, m)(1,5\pi s^{-1}) \cdot \sin\left((1,5\pi s^{-1})t + \frac{2}{3}\pi\right),$$

$$v(t) = -(0,09\pi\, m\ s^{-1}) \cdot \sin\left((1,5\pi s^{-1})t + \frac{2}{3}\pi\right)$$

while the acceleration by differentiating the velocity by time

$$a(t) = \frac{dv(t)}{dt} = v'(t) = -(0,09\pi\, ms^{-1})(1,5\pi s^{-1}) \cdot \cos\left((1,5\pi s^{-1})t + \frac{2}{3}\pi\right),$$

$$a(t) = -(0,135\pi^2\, ms^{-2}) \cdot \cos\left((1,5\pi s^{-1})t + \frac{2}{3}\pi\right).$$

Problem 5 ([8], s. 46) *Find the form of a wave equation whose solution is a plane wave propagating along the x-axis according to the equation*

$$u = u(x,t) = U \cdot \sin(\omega \cdot t - k \cdot x + \varphi).$$

Solution: Let us calculate the partial derivative of u by each variable

$$\frac{\partial u(x,t)}{\partial t} = U \cdot \omega \cdot \cos(\omega \cdot t - k \cdot x + \varphi),$$

$$\frac{\partial^2 u(x,t)}{\partial t^2} = -U \cdot \omega^2 \cdot \sin(\omega \cdot t - k \cdot x + \varphi) = -\omega^2 \cdot u(x,t),$$

$$\frac{\partial u(x,t)}{\partial x} = -U \cdot k \cdot \cos(\omega \cdot t - k \cdot x + \varphi),$$

$$\frac{\partial^2 u(x,t)}{\partial x^2} = -U \cdot k^2 \cdot \sin(\omega \cdot t - k \cdot x + \varphi) = -k^2 \cdot u(x,t).$$

Expressing u from the second partial derivative by t and substituting into the second partial derivative by x, we obtain

$$\frac{\partial^2 u}{\partial x^2} = \frac{k^2}{\omega^2} \cdot \frac{\partial^2 u}{\partial t^2},$$

where $c = \omega/k$ defines the phase velocity c. This results in the equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2} \cdot \frac{\partial^2 u}{\partial t^2}.$$

Problem 6 ([9], p. 53) *Filling and emptying a prismatic vessel through an opening with $Q_p = const.$*

Solution: If the vessel has a constant section of $A_z = const.$ (cylinder, prism, ...), we can write, for the time needed to change the level position from z_1 to z_2,

$$t = A_z \int_{z_1}^{z_2} \frac{dz}{Q_p - \mu A \sqrt{2gz}}, \tag{1.1}$$

where the inflow Q_p can be expressed using the area of the outflow opening A and the value z_u, at which the level would stabilize if the inflow was equal to the outflow:

$$Q_p = \mu A \sqrt{2gz_u},$$

where A is the area of the outflow opening and μ is an outflow coefficient. Substituting this into the integral (1.1), we obtain:

$$t = A_z \int_{z_1}^{z_2} \frac{dz}{\mu A \sqrt{2gz_u} - \mu A \sqrt{2gz}} = \frac{A_z}{\mu A \sqrt{2g}} \int_{z_1}^{z_2} \frac{dz}{\sqrt{z_u} - \sqrt{z}} \, .$$

To solve the integral, we can use the substitution:

$$y = \sqrt{z_u} - \sqrt{z}, \quad dz = -2(\sqrt{z_u} - y)\, dy$$

with the integration limits being:

z	z_1	z_2
$y = \sqrt{z_u} - \sqrt{z}$	$y_1 = \sqrt{z_u} - \sqrt{z_1}$	$y_2 = \sqrt{z_u} - \sqrt{z_2}$

The solution has the form:

$$t = -\frac{2A_z}{\mu A\sqrt{2g}} \int_{y(z_1)}^{y(z_2)} \frac{\sqrt{z_u} - y}{y}\, dy = \frac{2A_z}{\mu A\sqrt{2g}} \int_{y_1}^{y_2} \left(1 - \frac{\sqrt{z_u}}{y}\right) dy =$$

$$= \frac{2A_z}{\mu A\sqrt{2g}} \left[y - \sqrt{z_u} \ln|y|\right]_{\sqrt{z_u}-\sqrt{z_1}}^{\sqrt{z_u}-\sqrt{z_2}} = \frac{2A_z}{\mu A\sqrt{2g}} \left[\sqrt{z_1} - \sqrt{z_2} - \sqrt{z_u} \cdot \ln \frac{\sqrt{z_u} - \sqrt{z_2}}{\sqrt{z_u} - \sqrt{z_1}}\right].$$

For a prismatic vessel with no inflow ($z_u = 0$), the time needed to change the water level position from z_1 to z_2:

$$t = -A_z \int_{z_1}^{z_2} \frac{dz}{\mu A\sqrt{2gz}} = \frac{A_z}{\mu A\sqrt{2g}} \int_{z_2}^{z_1} \frac{1}{\sqrt{z}}\, dz =$$

$$= \frac{2A_z}{\mu A\sqrt{2g}} \left[\sqrt{z}\right]_{z_2}^{z_1} = \frac{2A_z}{\mu A\sqrt{2g}}(\sqrt{z_1} - \sqrt{z_2}).$$

The time needed to completely empty ($z_2 = 0$) a prismatic vessel without inflow ($z_u = 0$) is:

$$T = \frac{2A_z\sqrt{z_1}}{\mu A\sqrt{2g}}.$$

Problem 7 ([9], s. 54) *Filling a cylinder cistern with $Q_p = const$.*

Solution: A cylinder vessel with horizontal axis is being emptied through a hole at the lowest point while air flows into the space above the liquid level. The level area changes with the level $A_z = 2L\sqrt{z(2r - z)}$, where L is the length of the cistern, r is its radius and z is the level height. The time needed to change the level height from z_1 to z_2 is obtained by substituting A_z into (1.1) with $Q_p = 0$:

$$t = -\int_{z_1}^{z_2} \frac{A_z dz}{\mu A\sqrt{2gz}} = -\frac{2L}{\mu A\sqrt{2g}} \int_{z_1}^{z_2} \sqrt{2r - z}\, dz.$$

This integral can be solved by the substitution

$$y = 2r - z, \quad dz = -dy$$

with the integration limits:

z	z_1	z_2
$y = 2r - z$	$y_1 = 2r - z_1$	$y_2 = 2r - z_2$

The time needed to move the level from z_1 to z_2 is then:

$$t = \frac{2L}{\mu A\sqrt{2g}} \int_{y_1}^{y_2} \sqrt{y}\, dy = \frac{2}{3}\frac{2L}{\mu A\sqrt{2g}} \left[y^{\frac{3}{2}}\right]_{2r-z_1}^{2r-z_2} =$$

$$= \frac{4L}{3\mu A\sqrt{2g}} \left((2r-z_2)^{\frac{3}{2}} - (2r-z_1)^{\frac{3}{2}}\right).$$

The time during which the entire cistern ($z_1 = 2r = d$ a $z_2 = 0$) is emptied is

$$T = \frac{4Ld^{\frac{3}{2}}}{3\mu A\sqrt{2g}} \sim 0,301\frac{Ld^{\frac{3}{2}}}{\mu A}.$$

Problem 8 ([10], p. 38, Example 4.4) *Using the Planck law, derive the Stefan–Boltzman radiation law.*

Solution: The equation $\Phi_e = \int_0^\infty \Phi_\lambda\, d\lambda$, is transformed into a form containing radiation intensities (substituting $\Phi_e = M_e S$, and $\Phi_\lambda = M_\lambda S$), that is,

$$M_e = \int_0^\infty M_\lambda\, d\lambda$$

with the Planck law

$$M_\lambda(\lambda, T) = \frac{c_1}{\lambda^5 \cdot \left(e^{\frac{c_2}{\lambda T}} - 1\right)}, \quad c_1 = 2\pi hc^2,\ c_2 = \frac{hc}{k},$$

substituted for M_λ which yields the integral

$$M_e = \int_0^\infty \frac{c_1}{\lambda^5 \cdot \left(e^{\frac{c_2}{\lambda T}} - 1\right)}\, d\lambda\ .$$

Using the substitution $x = \frac{c_2}{\lambda T}$, we find

$$\lambda = \frac{c_2}{T}x^{-1}, \quad \frac{d\lambda}{dx} = -\frac{c_2}{T}x^{-2},$$

which is to be substituted into the integral equation. Then

$$M_e = \int_\infty^0 \frac{c_1}{\left(\frac{c_2}{T}x^{-1}\right)^5 (e^x - 1)} \left(-\frac{c_2}{T}x^{-2}\right) dx = \frac{c_1 T^4}{c_2^4} \int_0^\infty \frac{x^3}{e^x - 1}\, dx\,,$$

where the integral is

$$\int_0^\infty \frac{x^3}{e^x - 1}\, dx = \frac{\pi^4}{15}.$$

After substituting for the constants c_1 and c_2, we obtain the Stefan–Boltzman radiation law

$$M_e = \frac{2\pi^5 k^4}{15 h^3 c^2} T^4 = \sigma T^4.$$

Problem 9 ([10], p. 38, Example 4.5) *Using the Planck black-body radiation law, derive Wien's displacement law.*

Solution: The wave length in Wien's displacement law is subject to the maximum condition of the function $\dfrac{\partial M_\lambda}{\partial \lambda} = 0$. Substituting into this condition the Planck radiation law

$$M_\lambda(\lambda, T) = \frac{c_1}{\lambda^5 \cdot \left(e^{\frac{c_2}{\lambda T}} - 1\right)},$$

we get the equation

$$-\frac{5c_1}{\lambda^6} \cdot \frac{1}{e^{\frac{c_2}{\lambda T}} - 1} + \frac{c_1}{\lambda^5} \cdot \frac{\frac{c_2}{\lambda^2 T} e^{\frac{c_2}{\lambda T}}}{\left(e^{\frac{c_2}{\lambda T}} - 1\right)^2} = 0,$$

which can be cancelled by $\dfrac{c_1}{\lambda^6} \cdot \dfrac{1}{e^{\frac{c_2}{\lambda T}} - 1}$ yielding

$$\frac{c_2}{\lambda T} \cdot \frac{e^{\frac{c_2}{\lambda T}}}{e^{\frac{c_2}{\lambda T}} - 1} = 5.$$

By using the substitution $x = \frac{c_2}{\lambda T}$, we obtain the equation

$$x e^x = 5(e^x - 1),$$

which is solved by $x = 4,965$. Using the substitution $x = \frac{c_2}{\lambda T}$ we modify for the area of maximum radiation into the form $\lambda_m T = c_2/x = b$ where the wave length is denoted by the maximum index with the Wien constant $b = c_2/x$ introduced. After substituting the constant c_2 and $x = 4,965$, we get $b = 2,898 \cdot 10^{-3}\, m \cdot K$.

Problem 10 ([11], p. 119, Example 6.1) *Find the mean and virtual amperes of a harmonic current with a period of T and amplitude of I_m after half-wave rectification.*

Solution: The time dependence of the current $i = I_m \sin \omega t$ after half-wave rectification in each period is given by

$$i(t) = \begin{cases} I_m \sin \omega t & pro \quad t \in \langle 0, \frac{T}{2} \rangle, \\ 0 & pro \quad t \in \langle \frac{T}{2}, T \rangle, \end{cases}$$

where $\omega = 2\pi/T$.

a) The mean value of the current in the time period $\langle 0, T \rangle$ is

$$I_s = \frac{1}{T} \int_0^{T/2} I_m \sin \frac{2\pi}{T} t \, dt = \frac{1}{T} I_m \frac{T}{2\pi} \left[-\cos \frac{2\pi}{T} t \right]_0^{T/2} = \frac{I_m}{\pi}.$$

b) The virtual amperes I in the time period $\langle 0, T \rangle$ can be determined from the equation

$$I^2 = \frac{1}{T} \int_0^{T/2} I_m^2 \sin^2 \frac{2\pi}{T} t \, dt = \frac{1}{T} \frac{I_m^2}{2} \int_0^{T/2} \left(1 - \cos \frac{4\pi}{T} t \right) \, dt = \frac{I_m^2}{4},$$

$$I = \frac{I_m}{2}.$$

Chapter 2

Advanced problems

2.1 Euler's lateral stiffness

Determine the translation of all the points of a hinge-connected linearly elastic steel pole with a length of l, of a circular section with a diameter of $r << l$, loaded by an axial force of N; neglect gravitational forces. Young's positive modulus of elasticity E (in tension, compression, and torsion) is known.

Solution: Consider the pole as a one-dimensional structure with points $x \in \langle 0, l \rangle$, with a fixed joint at $x = 0$ and a telescopic joint at $x = l$ where a force N is applied, which is positive if compressive and negative if tensile

First calculate the area of the section A and the inertia moment J (for a planar problem, in a direction perpendicular to x):

$$A = \pi r^2 ,$$

$$\begin{aligned} J &= \int_0^{2\pi} \mathrm{d}\varphi \int_0^{r} (\rho \cos \varphi)^2 \cdot \rho \, \mathrm{d}\rho = \frac{1}{2} \int_0^{2\pi} (1 + \cos(2\varphi)) \, \mathrm{d}\varphi \int_0^{R} \rho^3 \, \mathrm{d}\rho \\ &= \frac{1}{2} \left[\varphi + \frac{1}{2} \sin(2\varphi) \right]_0^{2\pi} \left[\frac{1}{4} \rho^4 \right]_0^{r} = \frac{1}{2} \cdot 2\pi \cdot \frac{1}{4} r^4 = \frac{1}{4} \pi r^4 . \end{aligned}$$

Denote by σ the tension in the pole, by ε the deformation, by u the translation in the direction of x, and by w the translation in the direction perpendicular to x. In the first-order theory (equilibrium is formulated on a non-deformed structure) we have $w = 0$ with the stress and deformation conditions determined by Cauchy's equilibrium condition, Hooke's law, and by the relationship between translation and deformation:

$$\sigma' = 0 , \qquad \sigma = E\varepsilon , \qquad \varepsilon = u' .$$

The initial conditions are $u(0) = 0$ and $\sigma(l) = N/A$.

For the u variable, we only obtain

$$EAu'' = 0\,, \qquad u(0) = 0\,, \qquad EAu'(l) = N\,.$$

Thus the resulting translation is

$$u = \frac{N}{EA}x\,;$$

which corresponds to the constant deformation $\varepsilon = N/(EA)$ and stress $\sigma = N/A$.

This result is nearing reality only for $N < 0$ (tensile), or $N = 0$ (pole not loaded), but not for $N > 0$ (compressive). In the second-order theory (equilibrium is formulated on a deformed structure) also a bending moment $M = Nw$, is included, which can be written as

$$M = -EJ\kappa(w)\,,$$

where $\kappa(w)$ is the first curvature of the pole, that is, the deviation of its central line from the straight line in the osculating plane. Here, the boundary conditions are $w(0) = w(l) = 0$. Exactly, we have

$$\kappa(w) = \frac{w''}{\sqrt{(1 + w'^2)^{3/2}}}\,.$$

Thus, we obtain

$$\kappa(w) + \frac{N}{EJ}w = 0\,.$$

As yet, we have only considered $\kappa(w) = 0$, now put $\kappa(w) = w''$; this of course brings us to a special two-dimensional (planar) model. Denoting

$$\alpha = \sqrt{\frac{|N|}{EJ}}\,,$$

we get for $N < 0$ a linear differential equation with constant coefficients

$$w'' + \alpha^2 w = 0\,,$$

similarly, for $N > 0$,

$$w'' - \alpha^2 w = 0\,,$$

and finally, for $N = 0$, only $w'' = 0$, which means $w = 0$ due to the boundary conditions.

For $N < 0$, the general solution is

$$w = C_1 \sinh(\alpha x) + C_2 \cosh(\alpha x)\,,$$

where C_1 and C_2 are determined by the boundary conditions:

$$C_2 = 0\,, \qquad C_1 \sinh(\alpha l) + C_2 \cosh(\alpha l) = 0\,.$$

Since $\alpha l > 0$, $\sinh(\alpha l) = 0$ is not possible so that $C_1 = 0$, too, and $w = 0$. The result we have obtained is not anything new.

For $N > 0$, the general solution is

$$w = C_1 \sinh(\alpha x) + C_2 \cosh(\alpha x)\,,$$

where C_1 and C_2 are determined by the boundary conditions:

$$C_2 = 0\,, \qquad C_1 \sin(\alpha l) + C_2 \cos(\alpha l) = 0\,.$$

If $\sinh(\alpha l) \neq 0$, we have $C_1 = 0$ again, and so $w = 0$; the case $\sinh(\alpha l) \neq 0$, cannot, however, be excluded. This case occurs for $\alpha = k\pi/l$ and any natural k. Subsequently, we have

$$w = C_1 \sin \frac{k\pi x}{l}\,,$$

where C_1 can be chosen arbitrarily; for $x = l/2$, for instance, we have an arbitrary sag $w = C_1$. The corresponding compressive force is

$$N = EJ \left(\frac{k\pi}{l}\right)^2\,;$$

in practice, even the first force for $k = 1$, referred to as Euler's critical force, is dangerous. It is then usually used to derive what is called lateral stiffness used by many technical standards.

The non-realistic value of the bending w is due to the simplification: gravitation neglected, $\kappa(w)$ linearised, etc. For a more precise result, more background would be required from the theory of differential equations, particularly concerning the eigenvalues of differential operators, which may be non-linear, and their numeric solutions.

2.2 A beam on Winkler foundation

Determine the sag of a linearly elastic beam with a length of l fixed at both ends with a positive lateral uniform load of q, of a rectangular section with a length of b and height (in the direction of the load) of h. Young's bending modulus E for the beam is known to be positive with the beam placed on a Winkler foundation with a positive module (expressed as pressure per beam length unit) K. Investigate also the limit case of $K \to 0$.

Solution: First the inertia moment must be calculated

$$J = \int_{-b/2}^{b/2} \mathrm{d}x \int_{-h/2}^{h/2} y^2 \,\mathrm{d}y = [x]_{-b/2}^{b/2} \left[\frac{1}{3} y^3\right]_{-h/2}^{h/2} = \frac{1}{12} bh^3\,.$$

The equations of the elementary beam bending theory (at first without considering the effect of the foundation) for load q (constant here), translating force T, bending moment M, rotation φ and sag w (using the usual sign convention) are the following

$$T' = -q\,, \qquad M' = T\,, \qquad EJ\varphi' = -M\,, \qquad w' = \varphi\,;$$

excluding T, M, and φ yields

$$EJw'''' = q\,.$$

Directly integrating four times, we can easily get a general solution of this equation

$$EJw = C_0 + C_1x + C_2x^2 + C_3x^3 + \frac{qx^4}{24}\,.$$

For its first derivative, we then have

$$EJw' = C_1 + 2C_2x + 3C_3x^2 + \frac{qx^3}{6}\,.$$

The constants C_0, C_1, C_2, and C_3 can be calculated using the boundary conditions at the fixed points

$$w(0) = 0\,, \qquad \varphi(0) = 0\,, \qquad w(l) = 0\,, \qquad \varphi(l) = 0$$

or

$$w(0) = 0\,, \qquad w'(0) = 0\,, \qquad w(l) = 0\,, \qquad w'(l) = 0\,.$$

Clearly, $C_0 = C_1 = 0$; for the remaining two constants, we have the following system of equations

$$\begin{bmatrix} l^2 & l^3 \\ 2l & 3l^2 \end{bmatrix} \begin{bmatrix} C_2 \\ C_3 \end{bmatrix} = -\frac{ql^2}{24} \begin{bmatrix} l^2 \\ 4l \end{bmatrix},$$

which can be soved by Cramer's rule:

$$C_2 = -\frac{ql^2}{24} \cdot \frac{3l^4 - 4l^4}{3l^2 - 2l^2} = \frac{ql^2}{24}\,, \qquad C_3 = -\frac{ql^2}{24} \cdot \frac{4l^3 - 2l^3}{3l^2 - 2l^2} = -\frac{2ql}{24}\,.$$

Thus, we obtain

$$w = \frac{ql^2}{24EJ}(l-x)^2$$

and, particularly for $x = l/2$, the known formula

$$w(l/2) = \frac{ql^4}{384EJ}\,.$$

To include the foundation effect it is sufficient to replace q in the above differential equations by $q - kbw$. Denoting

$$\beta = \sqrt[4]{\frac{kb}{4EJ}}\,, \qquad \gamma = \sqrt[4]{\frac{q}{4EJ}}\,,$$

we get

$$4w'''' + \beta^4 w = \gamma^4\,.$$

The characteristic equation for the corresponding homogeneous equation (with the right-hand side equal to zero) is

$$4\lambda^4 + \beta^4 = 0\,.$$

Its four complex solutions in the form $\lambda = \rho(\cos\varphi + i\sin\varphi)$ can be found using de Moivre's theorem for $j \in \{0, 1, 2, 3\}$ and the equation

$$(\sqrt{2}\rho)^4(\cos(4\varphi) + i\sin(4\varphi)) = \beta^4(\cos((2j+1)\pi) + i\sin((2j+1)\pi))\,.$$

Clearly, $\sqrt{2}\rho = \beta$ and $\varphi = (2j+1)\pi/4$ so that

$$\lambda \in \{\beta + i\beta, \beta - i\beta, -\beta + i\beta, -\beta - i\beta\}\,.$$

A general solution of the (non-homogeneous) equation can then be written as

$$w = A_1 \exp(\beta x)\sin(\beta x) + A_2 \exp(\beta x)\cos(\beta x)$$

$$+ A_3 \exp(-\beta x)\sin(\beta x) + A_4 \exp(-\beta x)\cos(\beta x) + \gamma\,.$$

Its first derivative is

$$w' = (A_1 - A_2)\beta\exp(\beta x)\sin(\beta x) + (A_1 + A_2)\beta\exp(\beta x)\cos(\beta x)$$

$$-(A_3 + A_4)\beta\exp(-\beta x)\sin(\beta x) + (A_3 - A_4)\beta\exp(-\beta x)\cos(\beta x)\,.$$

The constants A_1, A_2, A_2, and A_4 can be calculated using the boundary conditions at the fixed points. Denoting for simplicity

$$\alpha_1 = \exp(\beta l)\sin(\beta l)\,, \qquad \alpha_2 = \exp(\beta l)\cos(\beta l)$$

$$\alpha_3 = \exp(-\beta l)\sin(\beta l)\,, \qquad \alpha_4 = \exp(-\beta l)\cos(\beta l)\,,$$

we obtain the following system of four equations

$$\begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & -1 \\ \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \\ \alpha_1 + \alpha_2 & \alpha_2 - \alpha_1 & \alpha_4 - \alpha_3 & -\alpha_3 - \alpha_4 \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{bmatrix} = \begin{bmatrix} \gamma \\ 0 \\ \gamma \\ 0 \end{bmatrix}.$$

The first two equations can be used to calculate

$$A_4 = \gamma - A_2\,, A_3 = A_4 - A_1 - A_2 = \gamma - A_1 - 2A_2\,;$$

substituting them will then yield a system of only two equations

$$\begin{bmatrix} \alpha_1 - \alpha_3 & \alpha_2 - 2\alpha_3 + \alpha_4 \\ \alpha_1 + \alpha_2 + \alpha_3 - \alpha_4 & \alpha_2 - \alpha_1 + 2\alpha_3 \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = \begin{bmatrix} \gamma(1 - \alpha_3 - \alpha_4) \\ 2\gamma\alpha_3 \end{bmatrix}.$$

By Cramer's rule then, $A_1 = D_1/D$ a $A_2 = D_2/D$ with

$$D = \alpha_1^2 - 5\alpha_1\alpha_3 + \alpha_2^2 - 2\alpha_2\alpha_4 + \alpha_3\alpha_4 - \alpha_1\alpha_4 + \alpha_4^2\,,$$

$$D_1 = -\gamma(\alpha_1 - \alpha_2 - 2\alpha_3 - \alpha_1\alpha_3 - \alpha_1\alpha_4 + 3\alpha_2\alpha_3 + \alpha_2\alpha_4 - 2\alpha_3^2)\,,$$
$$D_2 = \gamma(\alpha_1 + \alpha_2 + \alpha_3 - \alpha_4 - 3\alpha_1\alpha_3 - \alpha_1\alpha_4 - \alpha_2\alpha_3 - \alpha_2\alpha_4 + \alpha_3^2 + \alpha_4^2)\,.$$

Particularly, for $x = l/2$ (and the already known constants A_1, A_2, A_3 and A_4), we have

$$w(l/2) = A_1 \exp(\beta l/2) \sin(\beta l/2) + A_2 \exp(\beta l/2) \cos(\beta l/2)$$
$$+ A_3 \exp(-\beta l/2) \sin(-\beta l/2) + A_4 \exp(-\beta l/2) \cos(-\beta l/2) + \gamma$$

or (using familiar trigonometric formulas)

$$2w(l/2) = A_1(\exp(\beta l/2) - \alpha_2 \exp(-\beta l/2)) + A_2(\exp(\beta l/2) + \alpha_2 \exp(-\beta l/2))$$
$$+ A_3(\exp(-\beta l/2) - \alpha_4 \exp(\beta l/2)) + A_4(\exp(-\beta l/2) + \alpha_4 \exp(\beta l/2))) + 2\gamma\,.$$

The case of $k = 0$ has already been dealt with separately. For $k \to 0_+$ clearly $\beta \to 0_+$ and, using limit analysis (and L'Hospitals rule), we should achieve the same result. However, this would be too tedious and is, therefore, left to the patient reader.

2.3 Kirchhoff circular symmetric plate

Using the Kirchhoff theory of bending thin plates, determine the maximum sag of an elastic homogeneous and isotropic thin cylindrical plate with a thickness of h *and radius of* $r >> h$*, caused by a special rotational symmetric lateral load. The material density* ρ*, Young's elasticity modulus* E*, Poisson's constant* μ *(between* 0 *and* 1*) and the local gravitational acceleration* g *are known. The load as prescribed is such that its known line resultant* q *(force per length unit related to the plate diameter) does not depend on the distance from the centre of the plate.*

Solution: We will use cylindrical coordinates (ρ, φ, z) to describe the plate: the plate has the following position $\rho \in [0, r]$, $\omega \in [0, 2\pi)$, $z \in [-h/2, h/2]$. The normal stresses σ_r, σ_ω and tangential stress $\tau_{\rho z}$ are in Kirchhoff's theory for the rotational symmetric state of stress and deformation replaced by their line, force, and moment resultants, that is, by the translating force and bending moments in the radial and tangential directions

$$t = \int_{-h/2}^{h/2} \tau_{rz}\,\mathrm{d}z\,, \qquad m_\rho = \int_{-h/2}^{h/2} z\sigma_\rho\,\mathrm{d}z\,, \qquad m_\omega = \int_{-h/2}^{h/2} z\sigma_\omega\,\mathrm{d}z\,,$$

which means that the Cauchy conditions of static equilibrium only generate one force and one momentum equation (prime in this problem denotes the derivative by ρ)

$$t' + \frac{t}{\rho} + q = 0\,, \qquad m_\rho - m_\varphi + \rho m_\rho' = \rho t\,.$$

where q is the total lateral line load obtained by integrating over $\omega \in [0, 2\pi)$.

The first equation can be written in the form

$$\rho t' + t = -\rho q$$

to calculate t using the substitution $t = \exp(\eta)$. Clearly, $t' = exp(-\eta)\dot{t}$ (here, the dot denotes differentiation by the new variable η). After substituting, we already have an equation with constant coefficients

$$\dot{t} + t = \exp(-\eta)\,,$$

with a general solution

$$t = C\exp(-\eta) - \frac{q}{2}\exp(\eta) = \frac{C}{\rho} - \frac{q}{2}\rho$$

which depends on a constant C. However, since for $\rho \to 0_+$, $t \to \pm\infty$ is not possible, $C = 0$ is clearly true.

Translation in the direction of ρ is in Kirchhoff's theory considered as $u = -z\varphi$, where φ is the angle between the section of the middle plane of the deformed plate and the r axis, with $w' = \varphi$ for the sag w so that, for the deformation components, we have

$$\varepsilon_r = u' = -z\varphi'\,, \qquad \varepsilon_\varphi = \frac{u}{\rho} = -z\frac{\omega}{\rho}$$

and the linear Hooke law can be written in the form

$$\sigma_\rho = \frac{E}{1-\mu^2}(\varepsilon_r + \mu\varepsilon_\varphi)\,, \qquad \sigma_\omega = \frac{E}{1-\mu^2}(\varepsilon_\varphi + \mu\varepsilon_r)\,.$$

Expressing σ_r and σ_φ in terms of ε_ρ and ε_φ, and, subsequently, using φ and φ' in the result, introducing the concise denotation

$$B = \frac{Eh^3}{12(1-\mu^2)}$$

(referred to as lateral rigidity), after some simplification, we get

$$m_\rho = -B\left(\varphi' + \mu\frac{\varphi}{\rho}\right), \qquad m_\omega = -B\left(\mu\frac{\varphi}{\rho} + \mu\varphi'\right).$$

Substituting these equations into the moment equation, we finally obtain

$$\varphi'' + \frac{\varphi'}{\rho} - \frac{\varphi}{\rho^2} = -\frac{t}{B}.$$

As $t = -q\rho/2$, the derived equation can be written as

$$\rho^2\varphi'' + \rho\varphi' - \varphi = \frac{q}{2B}\rho^3$$

and used for calculating t by the substitution $\varphi = \exp(\kappa)$ (here, the dot denotes differentiating by the new variable κ). Clearly, $\varphi' = exp(-\kappa)\dot{\rho}$ and $\varphi' = exp(-2\kappa)(\ddot{\varphi} - \dot{\varphi})$. After substituting, we obtain an equation with constant coefficients

$$\ddot{\varphi} - \varphi = \frac{q}{2B}\exp(3\kappa),$$

with a general solution

$$\varphi = C_1\exp(\kappa) + C_0\exp(-\kappa) + \frac{q}{16B}\exp(3\kappa) = C_1\rho + \frac{C_0}{\rho} + \frac{q}{16B}\rho^3$$

which depends on the constants C_0 a C_1. However, since $\rho \to 0_+$ does not allow $\varphi \to \pm\infty$, clearly, $C_0 = 0$.

Using

$$\varphi = C_1\rho + \frac{q}{16B}\rho^3, \qquad \varphi' = C_1 + \frac{3q}{16B}\rho^2$$

we can now calculate

$$m_\rho = -B\left(C_1 + \frac{3q}{16B}\rho^2 + \mu C_1 + \mu\frac{q}{16B}\rho^2\right)$$

$$= -B\left((1+\mu)C_1 + (3+\mu)\frac{q}{16B}\rho^2\right).$$

Using the condition of zero lateral moment $m_\rho(r) = 0$, we then obtain

$$C_1 = -\frac{3+\mu}{1+\mu}\cdot\frac{q}{16B}r^2,$$

and thus

$$\varphi = -\frac{q}{16B}\left(\frac{3+\mu}{1+\mu}r^2\rho - \rho^3\right).$$

By direct integration, we calculate

$$w = -\frac{q}{16B}\left(\frac{3+\mu}{1+\mu}r^2\frac{\rho^2}{2} - \frac{\rho^4}{4}\right) + K = -\frac{q\rho^4}{64B}\left(2\frac{3+\mu}{1+\mu}r^2 - \rho^2\right) + K\,,$$

where the last unknown integration constant K can be found from the zero sag condition $w(r) = 0$ in the form

$$K = \frac{qr^4}{64B}\left(2\frac{3+\mu}{1+\mu} - 1\right) = \frac{3qr^4}{16Eh^3}\left(5 + 8\mu + 3\mu^2\right)\,.$$

Clearly, the equation $\varphi = 0$ (or $w' = 0$) has a single solution $\rho = 0$, which means that the function w can only reach extremal values at the points $\rho = 0$ and $\rho = r$. However, since $w(r) = 0$, the maximum sag of the plate is $w(0) = K$.

2.4 Fourier analysis of heat conduction through a wall

The temperature of the environment on the left-hand side of a wall with a thickness of l differs from a reference temperature τ at the time $t \in [0, t_*]$, where t_* is the length of a time interval given, by values given by a function $\vartheta_0(t)$, while that on the right-hand side by values given by a function $\vartheta_1(t)$; for $t = 0$, both functions are zeroes. Using a one-dimensional heat conduction model by Fourier's law, (ignoring internal heat sources, heat transfer and heat emission) describe the development of temperature T in the wall. Particularly, determine the temperature in the middle of the wall at the final time t_* assuming

$$\vartheta_1(t) = 0\,, \qquad \vartheta_0(t) = \frac{\delta t_*}{\pi}\sin\frac{\pi t}{t_*}\,,$$

where δ is the known time increment over time. The heat conductivity of the wall material λ, its thermal capacity (per unit of mass) c and density ρ are known.

Solution: Let $x \in [0, l]$ be the position of a point in the wall; here, the dot denotes differentiation by t and the prime by x. Then, the heat conduction equation for the unknown temperature $T(x, t)$ based on Fourier's law

$$q = -\lambda T'$$

for the heat flow $q(x, t)$, and on the energy conservation law

$$c\rho\dot{T} + q' = 0\,,$$

can be written in the form

$$c\rho\dot{T} - \lambda T'' = 0\,.$$

Introducing heat conductivity $a = \lambda/(c\rho)$ and denoting $\tau = T - \vartheta$ with

$$\vartheta(x,t) = \vartheta_0(t)\left(1 - \frac{x}{l}\right) + \vartheta_1(t)\frac{x}{l}\,,$$

we have

$$\dot{\tau} - a\tau'' = -\dot{\vartheta}\,,$$

where both for $x = 0$ and for $x = l$ always $\tau = 0$.

By the Fourier method of the separation of variables, τ can be assumed in the form

$$\tau = \sum_{n=1}^{\infty} \varphi_n(x)\psi_n(t)\,,$$

where the functions φ_n form an orthogonal basis of the space of Lebesgue square-integrable functions on the interval $[0,l]$, with $\varphi_n(0) = \varphi(l) = 0$, and the functions ψ_n belong to a similar space of functions on the interval $[0,t_*]$. Thus we have

$$\sum_{n=1}^{\infty} \varphi_n\dot{\psi}_n - a\sum_{n=1}^{\infty} \varphi_n''\psi_n = -\dot{\vartheta}\,.$$

In the first of the above spaces, we can define a dot product

$$(f,g) = \int_0^l f(x)g(x)\mathrm{d}x$$

for any two functions f and g. If, for any natural n, the equation is multiplied by a formal function $\varphi_n(x)$ and the result integrated over the interval $[0,l]$, the following result is obtained

$$(\varphi_n,\varphi_n)\dot{\psi}_n - a(\varphi_n,\varphi_n'')\psi_n = -(\varphi_n,\dot{\vartheta})\,.$$

Integrating by parts yields

$$(\varphi_n,\varphi_n)\dot{\psi}_n + a(\varphi_n',\varphi_n')\psi_n = -(\varphi_n,\dot{\vartheta})\,.$$

Thanks to the new denotation

$$\alpha_n = \frac{(\varphi_n',\varphi_n')}{(\varphi_n,\varphi_n)}\,, \qquad \zeta_n(t) = -\frac{(\varphi_n,\dot{\vartheta}(t))}{(\varphi_n,\varphi_n)}$$

this result can simply be written as

$$\dot{\psi}_n + \alpha_n a\psi_n = \zeta_n\,.$$

The equation thus derived can be solved for any natural n by the variation-of-constant method. A general solution of the homogeneous equation (with a zero right-hand side) is

$$\psi_n(t) = C_n\exp(-\alpha_n at)$$

for any real constant C_n; a particular solution of the original non-homogeneous equation can thus be found in the form

$$\psi_n(t) = K_n(t)\exp(-\alpha_n a t)\,,$$

which includes a certain time-dependent function K_n. However, the following is obvious

$$\dot{K}_n(t)\exp(-\alpha_n a t) = \zeta_n(t)\,,$$

and so, integrating, we obtain

$$K_n(t) = \int_0^t \exp(\alpha_n a s)\zeta_n(s)\,\mathrm{d}s\,.$$

The initial condition requiring τ to be zero for $t = 0$ clearly implies

$$\psi_n(t) = \int_0^t \exp(\alpha_n a(s-t))\zeta_n(s)\,\mathrm{d}s\,.$$

The classical Fourier sine series

$$\varphi_n(x) = \sin\frac{n\pi x}{l}$$

for natural n satisfies both the boundary conditions $\varphi_n(0) = \varphi_n(l)$, and the orthogonality conditions $(\varphi_n, \varphi_m) = 0$ for any natural m. Moreover, we have

$$(\varphi_n, \varphi_n) = \frac{l}{2}\,, \qquad (\varphi_n', \varphi_n') = \frac{(n\pi)^2}{2l}\,,$$

$$(\varphi_n, l) = -(-1)^n\frac{l^2}{n\pi}\,, \qquad (\varphi_n, l-x) = \frac{l^2}{n\pi}\,,$$

an thus also

$$\alpha_n = \left(\frac{n\pi}{l}\right)^2\,,$$

$$\zeta_n = \frac{2}{l}\left((\varphi_n, l-x)\frac{\dot{\vartheta}_0}{l} + (\varphi_n, x)\frac{\dot{\vartheta}_1}{h}\right) = \frac{1}{n\pi}\left(\dot{\vartheta}_0 - (-1)^n\dot{\vartheta}_1\right)\,.$$

(The derivation of these equations is not difficult but somewhat tedious; therefore we leave it to the reader as an integrating-by-parts exercise.) Since τ is a (theoretically infinite) sum of the products $\varphi_n(x)\psi_n(t)$, $\psi_n(x)$ is known and $\psi_n(t)$ can be calculated (using, for example, a numeric quadrature such as Simpson's method) from the integral containing the constant α_n and function $\zeta_n(s)$ for $s \in [0,t]$, we can already determine $T = \tau + \vartheta$ wherever in $[0,l]\times[0,t_*]$.

Denote $\nu = \pi/t_*$. Particularly, using the substitution $u = \nu t$, we have

$$\dot{\vartheta}_1 = 0\,, \qquad \dot{\vartheta}_0 = \cos u\,, \qquad \zeta_n = \frac{\delta}{n\pi}\cos u\,,$$

so that ψ_n can be obtained integrating by parts in an exact form

$$\psi_n = \frac{\delta\left(\alpha_n a \cos u + \nu \sin u - \alpha_n a \exp(-\alpha_n a t)\right)}{n\pi\left((\alpha_n a)^2 + \nu^2\right)} \,.$$

Also we have

$$\vartheta = \frac{\delta}{\nu} \sin u \,,$$

and thus

$$T = \sum_{n=1}^{\infty} \varphi_n \psi_n + \vartheta$$

$$= \frac{\delta}{\pi}\left(\sum_{n=1}^{\infty} \frac{1}{n} \cdot \frac{\alpha_n a \cos u + \nu \sin u - \alpha_n a \exp(-\alpha_n a t)}{(\alpha_n a)^2 + \nu^2} \sin \frac{n\pi x}{l} + t_* \sin u\right) \,.$$

Finally, we need to substitute $t = t_*$ and $x = l/2$. If we use the denotation ε_n so that $\varepsilon_n = 0$ for an even n, $\varepsilon_n = 1$ for an odd n such that $n + 1$ is divisible by four and $\varepsilon_n = -1$ for any other odd n, we obtain

$$\sin \frac{n\pi x}{l} = \sin \frac{n\pi}{2} = -\varepsilon_n \,;$$

moreover, $\sin u = \sin \pi = 0$ and $\cos u = \cos \pi = -1$. The resulting temperature is then (using the denotation α_n and ε_n) the sum of the infinite series

$$T(l/2, t_*) = \frac{\delta a}{\pi} \sum_{n=1}^{\infty} \frac{\alpha_n \varepsilon_n}{n} \cdot \frac{1 + \exp(-\alpha_n a t_*)}{(\alpha_n a)^2 + (\pi/t_*)^2} \,.$$

2.5 Bredt formulas for rod twisting

A thin-walled steel beam with a closed section, whose centre line is an ellipse with semi-axes a and $b \leq$ and and which has a constant thickness of $h << b$ is stressed with a positive twisting moment M in one third of its span $l >> a$ and, from two thirds to the end of its span, with a positive uniform twisting moment m, with $6M > ml$. At both ends, the beam is fixed to prevent twisting. Determine the extremal shear stress and extremal rotation of the beam assuming elastic deformation with a known steel elastic shear modulus of G.

Solution: In dealing with the twisting of a thin-walled rod with a constant thickness, we can apply a pair of Bredt formulas

$$W = 2Ah \,, \qquad I = \frac{4hA^2}{s} \,,$$

containing section modulus W, shear stiffness moment I, area of the section bounded by the centre curve A and length of this curve s. The extremal shear stress in the section (with the greatest absolute value) is then $\tau = K/W$ where K is the respective twisting moment, and the rotation of the section ϑ can, for $x \in [0, l]$, be obtained by solving the equation $K = -GI\varphi'$ with the initial conditions $\vartheta(0) = \vartheta(l) = 0$.

In our case, the centre curve is the ellipse $x = a\cos\varphi$, $y = b\sin\varphi$ for $\varphi \in [0, 2\pi)$, which means that

$$A = \int_0^{2\pi} \mathrm{d}\varphi \int_0^1 ab\rho\,\mathrm{d}\rho = 2\pi ab \left[\frac{1}{2}\rho^2\right]_0^1 = \pi ab\,.$$

Differentiating by φ (denoted by a dot), we get

$$\dot{x} = -a\sin\varphi, \quad \dot{y} = b\cos\varphi$$

and, using the denotation

$$k = \sqrt{1 - \left(\frac{b}{a}\right)^2},$$

subsequently,

$$s = \int_0^{2\pi} \sqrt{\dot{x}^2 + \dot{y}^2}\,\mathrm{d}\varphi = a\int_0^{2\pi} \sqrt{1 - (k\cos\varphi)^2}\,\mathrm{d}\varphi\,.$$

For $k = 0$, that is, for a circle with $a = b$, this elliptic integral has a well-known analytic result $s = 2\pi a = 2\pi b$, however, generally, $s \in [2\pi b, 2\pi a]$ can only be expressed as the sum of an infinite series or calculated using a numerical procedure.

Denote by M_0 the moment reaction at the left support next the intervals $\Omega_1 = \langle 0, l/3\rangle$, $\Omega_2 = \langle l/3, 2l/3)$, $\Omega_3 = \langle 2l/3, l\rangle$, $\Omega_1^- = \langle 0, l/3)$ and $\Omega_2^+ = (l/3, 2l/3\rangle$. The condition of static equilibrium yields the function of the twisting moment

$$K(x) = \begin{cases} -M_0 & \forall\, x \in \Omega_1^-\,, \\ M - M_0 & \forall\, x \in \Omega_2^+\,, \\ M - M_0 + m(x - 2l/3) & \forall\, x \in \Omega_3\,. \end{cases}$$

Note that, here, we cannot just replace Ω_1^- and Ω_2^+ by Ω_1 and Ω_2 as the limit of $K(x)$ for $x \to l/3$ does not exist (limits on the left and right are different). This shortcoming (which can only be resolved by what is referred to as Dirac distribution), however, will not prevent us from further calculation: in integrating, a single point has a zero measure.

As the problem is statically indeterminate, M_0 must be determined from the initial conditions. Considering the condition $\vartheta(0) = 0$, direct integration yields

$$-GI\vartheta(x) = \begin{cases} -M_0 x & \forall\, x \in \Omega_1\,, \\ -M_0 x + M(x - l/3) & \forall\, x \in \Omega_2\,, \\ -M_0 x + M(x - l/3) + m(x - 2l/3)^2 & \forall\, x \in \Omega_3\,. \end{cases}$$

From the condition $\vartheta(l) = 0$ (and thus also $-GI\vartheta(l) = 0$) we already have (independently of GI) the desired reaction

$$M_0 = \frac{2}{3}M + \frac{1}{18}ml\,.$$

Finally, we get the function of the twisting moment

$$K(x) = \begin{cases} -2M/3 - ml/18 & \forall\, x \in \Omega_1^-\,, \\ M/3 - ml/18 & \forall\, x \in \Omega_2^+\,, \\ M/3 - ml/18 + m(x - 2l/3) & \forall\, x \in \Omega_3\,, \end{cases}$$

while the rotation function is

$$GI\vartheta(x) = \begin{cases} (2M/3 + ml/18)x & \forall\, x \in \Omega_1\,, \\ (2M/3 + ml/18)x - M(x - l/3) & \forall\, x \in \Omega_2\,, \\ (2M/3 + ml/18)x - M(x - l/3) - m(x - 2l/3)^2/2 & \forall\, x \in \Omega_3\,. \end{cases}$$

The extremal stress τ will occur at the point of the extremal twisting moment K. The function K is clearly increasing so that

$$\min\tau = \frac{K(0)}{W} = -\frac{1}{W}\left(\frac{2}{3}M + \frac{1}{18}ml\right)\,,$$

$$\max\tau = \frac{K(l)}{W} = \frac{1}{W}\left(\frac{1}{3}M + \frac{5}{18}ml\right)\,.$$

Now we must find the extremal rotation, which necessitates $K = 0$ provided that the derivative of ϑ exists. Since $M > ml/6$, in Ω_1^- we clearly have $K < 0$ and in $\Omega_2^+ \cup \Omega_3$ we clearly have $K > 0$. Apart from $x = 0$ and $x = l$ where $\vartheta = 0$, we only have $x = l/3$ where

$$GI\vartheta(l/3) = \frac{l}{9}\left(2M + \frac{1}{6}ml\right)\,,$$

which means that

$$\min\vartheta = 0\,, \qquad \max\vartheta = \frac{1}{9GI}\left(2M + \frac{1}{6}ml\right)\,.$$

2.6 Boussinesq elastic foundation

Determine the subsidence at the depth h below ground, caused by single loads P taking effect at the vertices of a square with a diagonal of $2h$. Assume that the foundation behaves like a homogeneous isotropic elastic semi-space and that its positive modulus of elasticity $E > 0$ and the Poisson constant $\mu \in \langle 0, 1/2)$ are known.

Solution: For conciseness, denote $a = 1 - 2\mu$ a $b = a + 1$ and introduce the Lame constants

$$\lambda_2 = \frac{E}{1+\mu}\,, \qquad \lambda_1 = \frac{\lambda_2 \mu}{a}$$

(λ_2 coincides with the modulus of elasticity in shear) and the Cartesian coordinates $x_i,\ i \in \{1, 2, 3\}$. We will also assume that the load is exerted in a direction perpendicular to the plane $x_3 = 0$ at the points $A_1 = [h, 0, 0]$, $A_2 = [-h, 0, 0]$, $A_3 = [0, h, 0]$, $A_4 = [0, -h, 0]$.

For the stress components σ_{ij} in the elastic semi-space $x_3 \geq 0$, we have the Cauchy equilibrium equations

$$\sum_{k=1}^{3} \partial\sigma_{kj}/\partial x_j = 0 \qquad \forall\, j \in \{1, 2, 3\}\,,$$

for the deformation components ε_{ij} then Hooke's constitutive equation

$$\sigma_{ij} = \lambda_1 \delta_{ij} \sum_{k=1}^{3} \varepsilon_{kk} + 2\lambda_2 \varepsilon_{ij} \qquad \forall\, i, j \in \{1, 2, 3\}\,,$$

where the Kronecker symbol δ_{ij} equals 1 for $i = j$ and 0 for $i \neq j$ and, for the translation components with respect to the original geometric configuration (that is, the non-loaded semi-space) u_i, u_j we have

$$\varepsilon_{ij} = \frac{1}{2}\left(\partial u_i/\partial x_j + \partial u_j/\partial x_i\right) \qquad \forall\, i, j \in \{1, 2, 3\}\,.$$

Expressing each stress component in the Cauchy equilibrium equations in terms of the deformation components and these, in turn, in terms of the translation, we obtain a stationary equation of linear elasticity (without internal loads) in the following form

$$(\lambda_1 + \lambda_2) \sum_{k=1}^{3} \partial^2 u_k/\partial x_i \partial x_k + \lambda_1 \partial^2 u_i/\partial x_k^2 = 0\,.$$

The solution to this equation must comply with the initial conditions. In our case, for the lateral load $q(x_1, x_2, x_3)$ at the border $x_3 = 0$, we would obtain the equilibrium conditions

$$\sigma_{i1} = \sigma_{i2} = 0\,, \qquad \sigma_{i3} = q\,,$$

which could again be expressed in terms of the translation components, however, formally, q would be zero everywhere except at the points A_1, A_2, A_3 and A_4, while at these points, q would have to tend to infinity. Nevertheless, instead of P, the load $q = P/\omega$ can be considered in the plane $x_3 = 0$ in a sufficiently small region with an area of ω and the limit behaviour for $\omega \to \infty$ studied. A mathematically correct approach, however, would require familiarity with the concepts of the theory of distributions, in particular the Dirac distribution and Heaviside function, as well as more profound knowledge of the theory of partial differential equations and their systems. However, from the engineering point of view, the most important thing is that, for a case similar to ours with a single load having the components F_i, $i \in \{1,2,3\}$ that takes effect at the origin, we can employ what is termed a Boussinesq solution

$$u_i = \frac{1}{4\pi\lambda_2}\sum_{k=1}^{3} G_{ik}F_k \qquad \forall i \in \{1,2,3\}\,,$$

where

$$\begin{aligned}
G_{11} &= b/r + x_1^2/r^3 - ax_1^2/(r(r+x_3)^2) - ax_3/(r(r+x_3))\,,\\
G_{12} &= x_1x_2/r^3 - ax_1x_2/(r(r+x_3)^2)\,,\\
G_{13} &= x_1x_3/r^3 - ax_1/(r(r+x_3))\,,\\
G_{22} &= b/r + x_2^2/r^3 - ax_2^2/(r(r+x_3)^2) - ax_3/(r(r+x_3))\,,\\
G_{23} &= x_2x_3/r^3 - ax_2/(r(r+x_3))\,,\\
G_{33} &= b/r + x_3^2/r^3\,,
\end{aligned}$$

$r = \sqrt{x_1^2 + x_2^2 + x_3^2}$ a $G_{ij} = G_{ji}$ for all $i,j \in \{1,2,3\}$.

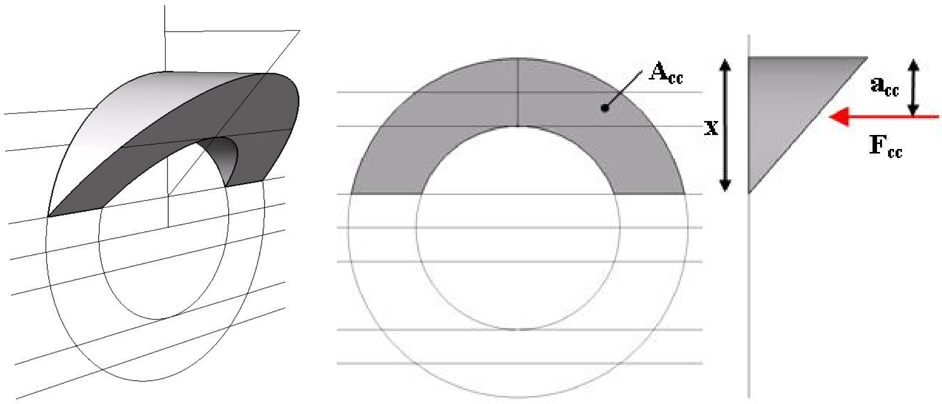
Because the problem is symmetric and linear (the respective differential operator is additive and homogeneous, which in engineering mechanics corresponds to principles of adequacy and superposition of effects), we can write $u_1(0,0,h) = u_2(0,0,h) = 0$. Then we can choose $x_1 = x_3 = h$, $x_2 = 0$, $F_1 = F_2 = 0$ and $F_3 = P$ four times only calculating $r = h\sqrt{2}$ and the subsidence

$$\begin{aligned}
u_3(0,0,h) &= \frac{P}{\pi\lambda_2}G_{33}(h,0,h) = \frac{P}{\pi\lambda_2}\left(\frac{b}{h\sqrt{2}} + \frac{h^2}{2\sqrt{2}\,h^3}\right)\\
&= \frac{b+\frac{1}{2}}{2\sqrt{2}\,\lambda_2}\cdot\frac{P}{\pi h} = \frac{(5-4\mu)(1+\mu)P}{4\sqrt{2}\,\pi E h}\,.
\end{aligned}$$

2.7 The volume and centre of mass of the pressed concrete area

When concrete structures are sized, problems arise with determining the force F_{cc} in the pressed area of a concrete section. This force is, for instance, determined for what is called the limit state of applicability assuming a linear stress distribution σ_c. In the solution, the position of the neutral (zero) axis x, must be known, that is, the axis with a zero relative deformation ε_c. This position *can be determined by solving the conditions of the equilibrium of the external and internal forces (force and moment equilibrium condition) either directly or by iteration. It is in this process that the problem occurs of determining the force F_{cc} as the value of stress σ_c in the pressed concrete region A_{cc}, whose shape can have an entirely general form, as well as determining the position of the force F_{cc} in the pressed section region a_{cc}, as the centre of the stress volume σ_c.*

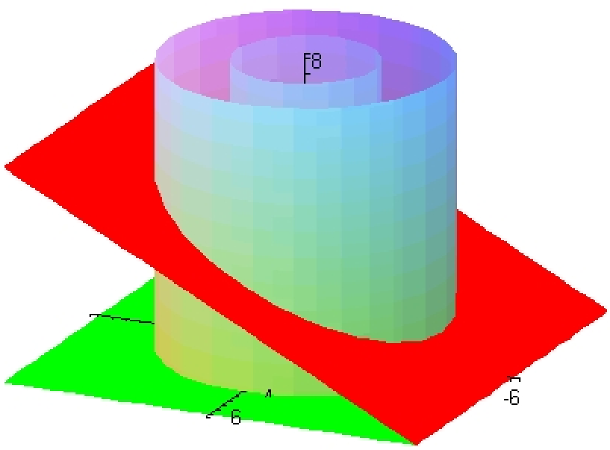
For this particular problem, the section has been chosen in the shape of a circular ring.



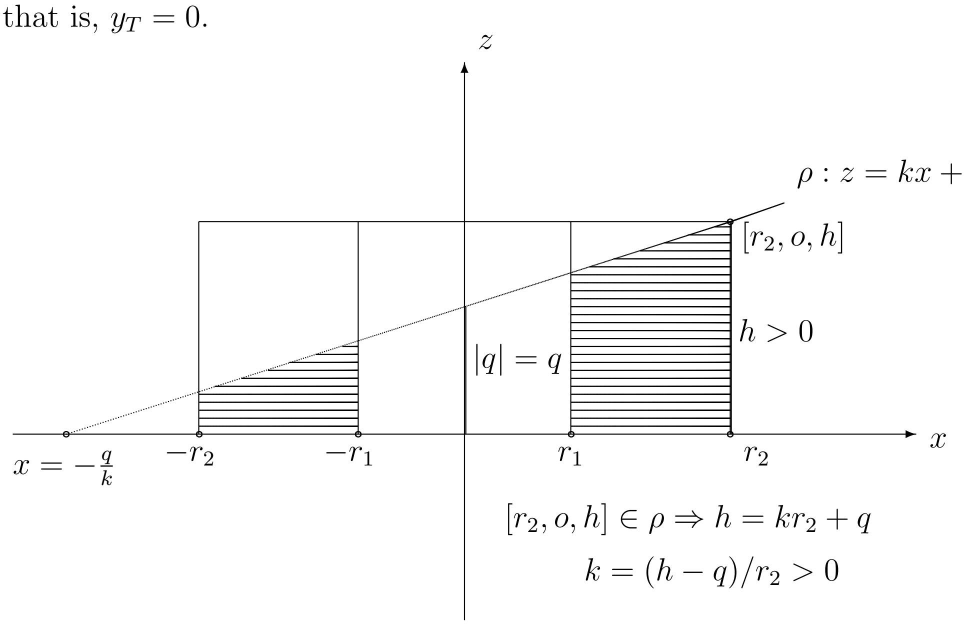
Solution:
The object can be positioned in the system of coordinates so that
- the section plane is parallel, say, to the y-axis
- the z-axis coincides with the axis of the circular ring,

that is,

$$r_1^2 \leq x^2 + y^2 \leq r_2^2,\ 0 \leq z \leq kx + q.$$



The cutting plane passing through the point $[r_2, 0, h]$ will then be defined by the equation $z = kx + q$, $k > 0$ intersecting the xy-plane in the straight-line $x = -q/k$. The centre of mass of the homogeneous object will lie in the xz-plane, that is, $y_T = 0$.



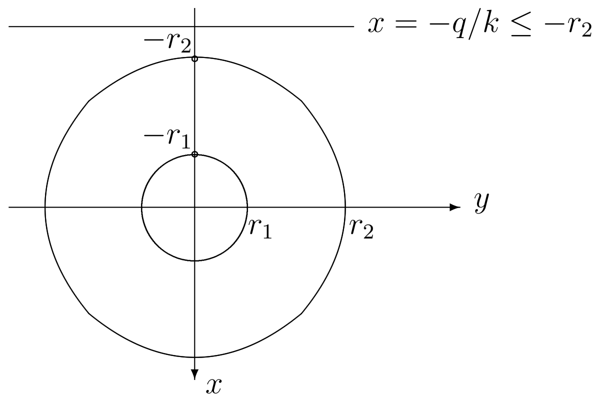
We will investigate four section types:

- (A) $-q/k < -r_2$, (B) $r_1 \leq -q/k < r_2$,
- (C) $-r_2 \leq -q/k \leq -r_1$, (D) $-r_1 \leq -q/k \leq r_1$.

(A) Type

$$-q/k < -r_2 \Leftrightarrow q \geq kr_2 = h - q \Leftrightarrow q > \frac{h}{2},\ q \leq h$$

is the simplest one as the integration domain includes the entire circular ring, which makes it possible to apply the cylindrical coordinates.



The density of a homogeneous body being $\rho(x, y, z) \equiv c$, we can put $c = 1$. without loss of generality.
The integration domain

$$\Omega: \quad r_1^2 \leq x^2 + y^2 \leq r_2^2, \quad 0 \leq z \leq kx + q$$

through

$$\begin{array}{l} x = \rho \cos\varphi \\ x = \rho \sin\varphi \\ z = z \\ \hline |J(\rho, \varphi)| = \rho \end{array}$$

changes into

$$\Omega_1: \quad r_1 \leq \rho \leq r_2, \quad -\pi \leq \varphi \leq \pi, \quad 0 \leq z \leq k\rho\cos\varphi + q.$$

Then, for the mass (numerically, also the volume), we have

$$m(\Omega) = \iiint_\Omega dxdydz = \iiint_{\Omega_1} |J(\rho, \varphi)| d\rho d\varphi dz =$$

$$= \int_{r_1}^{r_2} \rho d\rho \int_{-pi}^{\pi} d\varphi \int_0^{k\rho\cos\varphi + q} dz = q\pi \cdot (r_2^2 - r_1^2),$$

and, for the static moments with respect to the coordinate axes,

$$S_{yz}(\Omega) = \iiint_\Omega x \; dxdydz = \frac{1}{4} k\pi \cdot (r_2^2 - r_1^2)(r_2^2 + r_1^2),$$

$$S_{xz}(\Omega) = \iiint_\Omega y \; dxdydz = 0,$$

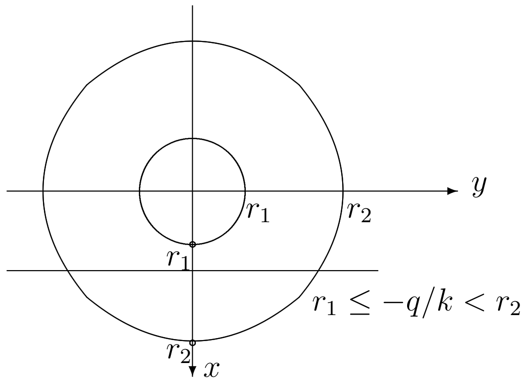
$$S_{xy}(\Omega) = \iiint_\Omega z \; dxdydz = \frac{\pi}{8} \cdot (r_2^2 - r_1^2) \cdot (k^2(r_1^2 + r_2^2) + 4q^2).$$

Therefore, the resulting centre of mass has the following coordinates

$$T = \left[\frac{S_{yz}(\Omega)}{m(\Omega)}, \frac{S_{xz}(\Omega)}{m(\Omega)}, \frac{S_{xy}(\Omega)}{m(\Omega)} \right] = \left[\frac{k}{4q}(r_1^2 + r_2^2), 0, \frac{k^2}{8q}(r_1^2 + r_2^2) + \frac{q}{2} \right]$$

for $q/k \geq r_2$ (tj., $q > h/2$).

(B) We will investigate the case $r_1 \leq -q/k < r_2$.



For a constant $a > 0$, first define functions F_0, F_1, F_2 to simplify the notation of the integration results:

$$
\begin{aligned}
&F_0(x,a) = \tfrac{1}{2}x\sqrt{a^2-x^2} + \tfrac{1}{2}a^2 \operatorname{arctg}\tfrac{x}{\sqrt{a^2-x^2}} \ \left(+const = \textstyle\int \sqrt{a^2-x^2}\,dx\right),\\
&F_1(x,a) = -\tfrac{1}{3}(\sqrt{a^2-x^2})^3 \ \left(+const = \textstyle\int x\sqrt{a^2-x^2}\,dx\right),\\
&F_2(x,a) = -\tfrac{1}{4}x(\sqrt{a^2-x^2})^3 + \tfrac{1}{8}a^2x\sqrt{a^2-x^2} + \tfrac{1}{8}a^4 \operatorname{arctg}\tfrac{x}{\sqrt{a^2-x^2}}\\
&\left(+const = \textstyle\int x^2\sqrt{a^2-x^2}\,dx\right).
\end{aligned}
\tag{2.1}
$$

Then the conditions

$$
\begin{array}{ll}
F_0(-x,a) = -F_0(x,a), & F_0(a,a) = \frac{1}{4}\pi a^2 = -F_0(-a,a),\\
F_1(-x,a) = F_1(x,a), & F_1(a,a) = 0 = F_1(-a,a),\\
F_2(-x,a) = -F_1(x,a), & F_2(a,a) = \frac{1}{16}\pi a^4 = -F_2(-a,a).
\end{array}
\tag{2.2}
$$

are fulfilled. Now, the integration domain is

$$\Omega : \quad -\frac{q}{k} \le x \le r_2,\ -\sqrt{r_2^2-x^2} \le y \le \sqrt{r_2^2-x^2},\ 0 \le z \le kx+q.$$

Calculating, we obtain

$$m(\Omega) = \iiint_\Omega dxdydz = \int_{-q/k}^{r_2} dx \int_{-\sqrt{r_2^2-x^2}}^{\sqrt{r_2^2-x^2}} dy \int_0^{kx+q} dz =$$

$$= 2\int_{-q/k}^{r_2} (kx+q)\sqrt{r_2^2-x^2}dx = [2kF_1(x,r_2) + 2qF_0(x,r_2)]_{-q/k}^{r_2} =$$

$$= \frac{1}{2}\pi q r^2 - 2kF_1(\frac{q}{k}, r_2) + 2qF_0(\frac{q}{k}, r_2),$$

$$S_{yz}(\Omega) = \iiint_\Omega x\ dxdydz = 2\int_{-q/k}^{r_2} x(kx+q)\sqrt{r_2^2-x^2}dx =$$

$$= [2kF_2(x, r_2) + 2qF_1(x, r_2)]_{-q/k}^{r_2} = \frac{1}{8}k\pi r_2^4 + 2kF_2(\frac{q}{k}, r_2) - 2qF_1(\frac{q}{k}, r_2),$$

$$S_{xz}(\Omega) = \iiint_\Omega y \; dxdydz = 0,$$

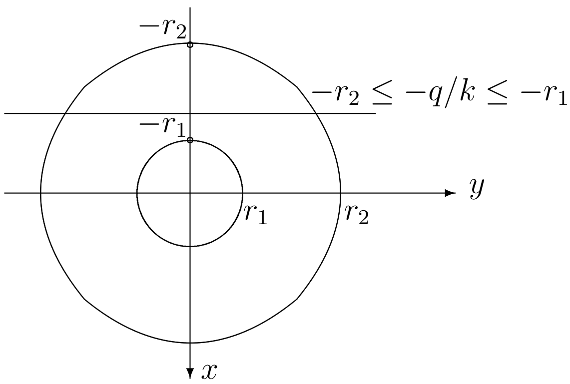
$$S_{xy}(\Omega) = \iiint_\Omega z \; dxdydz = \int_{-q/k}^{r_2} (kx^2 + 2kqx + q^2)\sqrt{r_2^2 - x^2}dx =$$

$$= \left[k^2F_2(x, r_2) + 2kqF_1(x, r_2) + q^2F_0(x, r_2)\right]_{-q/k}^{r_2} =$$

$$= \frac{1}{16}\pi r_2^2(k^2r_2^2 + 4q^2) + q^2F_0(\frac{q}{k}, r_2) - 2kqF_1(\frac{q}{k}, r_2) + k^2F_2(\frac{q}{k}, r_2).$$

To determine the centre of mass coordinates in this and subsequent cases, suitable computer programs are needed with a spreadsheet such as Excel being sufficient.

(C) The case $-r_2 \le -q/k \le -r_1$ can be solved by combining the previous procedures using the difference of integrals.



The integration domain $\Omega = \Omega_1 - \Omega_2$ where

$$\Omega_1 : \quad r_1^2 \le x^2 + y^2 \le r_2^2, \quad 0 \le z \le kx + q,$$

$$\Omega_2 : \quad -r_2 \le x \le -\frac{q}{k}, \; -\sqrt{r_2^2 - x^2} \le y \le \sqrt{r_2^2 - x^2}, \; 0 \le z \le kx + q.$$

Then

$$m(\Omega) = m(\Omega_1) - \iiint_{\Omega_2} dxdydz =$$

$$= q\pi(r_2^2 - r_1^2) + [2kF_1(x, r_2) + 2qF_0(x, r_2)]_{-q/k}^{-r_2} =$$

$$= q\pi(r_2^2 - r_1^2) - \frac{1}{2}q\pi r_2^2 + 2qF_0(\frac{q}{k}, r_2) - 2kF_1(\frac{q}{k}, r_2)$$

and, similarly,

$$S_{yz}(\Omega) = \frac{1}{4}k\pi \cdot (r_2^4 - r_1^4) + [2kF_2(x, r_2) + 2qF_1(x, r_2)]_{-q/k}^{-r_2} =$$

$$= \frac{1}{4}k\pi \cdot (r_2^4 - r_1^4) - \frac{1}{8}k\pi r_2^4 + 2kF_2(\frac{q}{k}, r_2) - 2qF_1(\frac{q}{k}, r_2),$$

$$S_{xz}(\Omega) = 0,$$

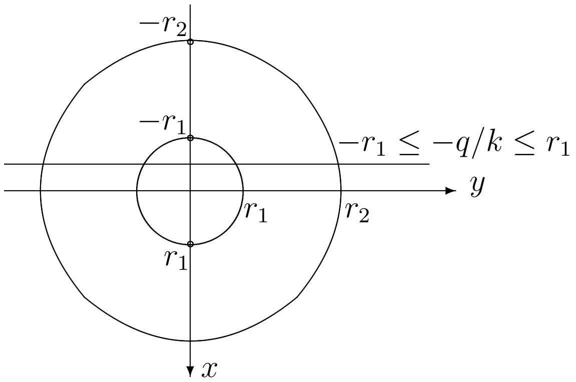
$$S_{xy}(\Omega) = \frac{\pi}{8} \cdot (r_2^2 - r_1^2) \cdot (k^2(r_1^2 + r_2^2) + 4q^2) +$$

$$+ \left[k^2F_2(x, r_2) + 2kqF_1(x, r_2) + q^2F_0(x, r_2)\right]_{-q/k}^{-r_2} =$$

$$= \frac{\pi}{8} \cdot (r_2^2 - r_1^2) \cdot (k^2(r_1^2 + r_2^2) + 4q^2) - \frac{1}{16}\pi k^2 r_2^4 - \frac{1}{4}\pi q^2 r_2^2 +$$

$$+ k^2F_2(\frac{q}{k}, r_2) - 2kqF_1(\frac{q}{k}, r_2) + q^2F_0(\frac{q}{k}, r_2).$$

(D) The last case $-r_1 \leq -q/k \leq r_1$



can be solved by integrating over $\Omega = \Omega_1 - \Omega_2$ where

$$\Omega_1: \quad -\frac{q}{k} \leq x \leq r_2, \ -\sqrt{r_2^2 - x^2} \leq y \leq \sqrt{r_2^2 - x^2}, \ 0 \leq z \leq kx + q,$$

$$\Omega_2: \quad -\frac{q}{k} \leq x \leq r_1, \ -\sqrt{r_2^2 - x^2} \leq y \leq \sqrt{r_2^2 - x^2}, \ 0 \leq z \leq kx + q,$$

with the following results:

$$m(\Omega) = \iiint_{\Omega_1} dxdydz - \iiint_{\Omega_2} dxdydz =$$

$$
\begin{aligned}
&= 2\int_{-q/k}^{r_2} (kx+q)\sqrt{r_2^2 - x^2}dx - 2\int_{-q/k}^{r_1} (kx+q)\sqrt{r_1^2 - x^2}dx = \\
&= [2kF_1(x,r_2) + 2qF_0(x,r_2)]_{-q/k}^{r_2} - [2kF_1(x,r_1) + 2qF_0(x,r_1)]_{-q/k}^{r_1} = \\
&= \frac{1}{2}\pi q(r_2^2 - r_1^2) + 2q\left(F_0(\frac{q}{k}, r_2) - F_0(\frac{q}{k}, r_1)\right) - 2k\left(F_1(\frac{q}{k}, r_2) - F_1(\frac{q}{k}, r_1)\right),
\end{aligned}
$$

$$
\begin{aligned}
S_{yz}(\Omega) &= [2kF_2(x,r_2) + 2qF_1(x,r_2)]_{-q/k}^{r_2} - [2kF_2(x,r_1) + 2qF_1(x,r_1)]_{-q/k}^{r_1} = \\
&= \frac{1}{8}k\pi(r_2^4 - r_1^4) + 2k\left(F_2(\frac{q}{k}, r_2) - F_2(\frac{q}{k}, r_1)\right) - 2q\left(F_1(\frac{q}{k}, r_2) - F_1(\frac{q}{k}, r_1)\right),
\end{aligned}
$$

$$S_{yz}(\Omega) = 0,$$

$$
\begin{aligned}
S_{xy}(\Omega) &= \left[k^2F_2(x,r_2) + 2kqF_1(x,r_2) + q^2F_0(x,r_2)\right]_{-q/k}^{r_2} - \\
&- \left[k^2F_2(x,r_1) + 2kqF_1(x,r_1) + q^2F_0(x,r_1)\right]_{-q/k}^{r_1} = \\
&= \frac{1}{16}k^2\pi(r_2^4 - r_1^4) + \frac{1}{4}\pi q^2(r_2^2 - r_1^2) + k^2\left(F_2(\frac{q}{k}, r_2) - F_2(\frac{q}{k}, r_1)\right) - \\
&-2kq\left(F_1(\frac{q}{k}, r_2) - F_1(\frac{q}{k}, r_1)\right) + q^2\left(F_0(\frac{q}{k}, r_2) - F_0(\frac{q}{k}, r_1)\right).
\end{aligned}
$$

Thus, we have described all the types providing the necessary formulas.

2.8 Optimizing concrete mix by Feret

Design a procedure for the maximum-saving concrete-mixture design to be used for a concrete column with a constant section a given height of l that is axially stressed by a total compressive force of N, knowing the local gravitational acceleration g. For calculating the strength, use the empiric Feret formula with a known strength constant. Consider that the concrete density is linearly dependent on the strength. Assume that the concrete volume unit price α and the aggregate volume unit price β (taken to be the average of all the fractions used in a constant ratio) are known; ignoring the price of water and air. Prescribed are the limits c_1 and c_2 $(0 < c_1 < c_2 < 1)$ for the cement content per mixture unit and the limits ν_1 and ν_2 $(0 < \nu_1 < \nu_2)$ for the volume proportion of water and cement per mixture unit, guaranteeing that the column will bear more than its own weight.

Solution: Denote by c, s and w numbers from the interval $\langle 0, 1\rangle$, indicating the cement, aggregate, and water (including steam and other gases) content respectively per mixture volume unit; clearly, $c + s + w = 1$. The price of the mixture needed for a unit of the column height is

$$P = (\alpha c + \beta s)A,$$

(the total mixture price totalling to Pl) where A is the (not previously known) area of the column section. The compressive stress σ in the column linearly changes with its height; obviously, we have

$$\frac{N}{A} \leq \sigma \leq \frac{N}{A} + \rho g l \, .$$

Denote $\nu = w/c$. Knowing the positive strength constant ψ, by Feret, we can calculate the resulting concrete compressive strength

$$\kappa = \frac{\psi}{(1+\nu)^2} \, .$$

Generally, we require $\sigma \leq \kappa$; so the most economical option is

$$\kappa = \frac{N}{A} + \rho g l \, .$$

As we know the linear dependence of ρ on κ, we can write with economy

$$\rho g l = \gamma_0 + \gamma_1 \kappa \, ,$$

where γ_0 and γ_1 are known constants always positive in practice. For the column to bear more than its own height, $\kappa > gl\rho$ must be true and thus

$$\frac{(1-\gamma_1)\psi}{N(1+\nu_2)^2} - \frac{\gamma_0}{N} > 0 \, ;$$

then, clearly, $\gamma_1 < 1$ and, with the denotation

$$\lambda_0 = \frac{\gamma_0}{N} \, , \qquad \lambda_1 = \frac{(1-\gamma_1)\psi}{N}$$

also

$$\lambda_0 \leq \frac{\lambda_1}{(1+\nu_2)^2}$$

with a simple imlication $\lambda_0 < \lambda_1$.

For the actual load, we get

$$\frac{N}{A} = \frac{\psi}{(1+\nu)^2} - \rho g l = \frac{\psi}{(1+\nu)^2} - \rho g l = \frac{(1-\gamma_1)\psi}{(1+\nu)^2} - \gamma_0 \, ,$$

which implies

$$A = N \left(\frac{(1-\gamma_1)\psi}{(1+\nu)^2} - \gamma_0 \right)^{-1} = \left(\frac{\lambda_1}{(1+\nu)^2} - \lambda_0 \right)^{-1} .$$

Since $s = 1 - c - w$ and $w = c\nu$, we finally obtain

$$P = (\alpha c + \beta s) Q^{-1} = (\beta + (\alpha - \beta)c - \beta c \nu) Q^{-1} \, ,$$

with a concise denotation

$$Q = \frac{\lambda_1}{(1+\nu)^2} - \lambda_0\,,$$

as a function of two variables c and ν on the domain $\Omega = \langle c_1, c_2\rangle \times \langle \nu_1, \nu_2\rangle$. Its partial derivatives $\partial P/\partial c$ and $\partial P/\partial \nu$ can be written in the form (arranged in a way suitable for further investigation)

$$Q\,\partial P/\partial c = \alpha - \beta - \beta\nu\,,$$

$$\frac{Q(1+\nu^3)}{\beta}\,\partial P/\partial \nu = \left(\lambda_0\nu^3 + 3\lambda_0\nu^2 + 3(\lambda_0 - \lambda_1)\nu + (\lambda_0 - 3\lambda_1 + 2\lambda_1\frac{\alpha}{\beta}\right) c + 2\lambda_1\,.$$

If we put $\partial P/\partial c = 0$, we can use the frst equation to calculate c, and, if we put $\partial P/\partial \nu = 0$, we can use the first equation to also calculate ν. Thus the function P has a single stationary point $A_* = [c_*, \nu_*]$ (provided that it falls into Ω)

$$c_* = \frac{2\lambda_1}{\lambda_0 + \lambda_1(2\xi - 3) + 3(\lambda_1 - \lambda_0)(\xi - 1) - 3\lambda_0(\xi-1)^2(\xi+2)}\,,$$

$$\nu_* = \xi - 1\,.$$

However, at the stationary point $A_* = [c_*, \nu_*]$ thus obtained, the function P does not have to achieve its absolute minimum. The vertices $A_1 = [c_1, \nu_1]$, $A_2 = [c_1, \nu_2]$, $A_3 = [c_2, \nu_1]$, $A_4 = [c_2, \nu_2]$ and the stationary points of the line segments A_1A_2 and A_3A_4 have to be considered separately; not on the line segments A_2A_3 and A_1A_4 because, here, P is only a linear function of c. Non-trivial is only the analysis of the behaviour of P for $c = c_i$, $i \in \{1, 2\}$ where we have

$$\left(\lambda_0\nu^3 + 3\lambda_0\nu^2 + 3(\lambda_0 - \lambda_1)\nu + \lambda_0 - 3\lambda_1 + 2\lambda_1\xi\right) c_i + 2\lambda_1 = 0\,,$$

which is a cubic algebraic equation to be used to calculate (one to three) solutions ν. In the special case $\lambda_0 = 0$ (that is, $\gamma_0 = 0$) we get a unique solution

$$\nu_{i*} = \frac{2}{c_i} - 3 + 2\xi\,;$$

so next we can only work with $\lambda_0 > 0$ introducing the notation $\eta = \lambda_1/\lambda_0$ and rewriting the cubic equation as

$$\nu^3 + 3\nu^2 + 3(1-\eta)\nu + 1 - 3\eta + 2\eta\xi + \frac{\xi}{c_i} = 0\,.$$

The cubstitution $\mu = \nu - 1$ then yields

$$\mu^3 - 3\eta\mu + 2\eta\left(\xi + \frac{1}{c_i}\right) = 0\,.$$

The three potential solutions (some of them may not be real or belong to the interval $\langle \nu_1, \nu_2 \rangle$) can be obtained using the Cardano formula $\mu_{i*} = u + v$, $\mu_{i*} = \varepsilon_1 u + \varepsilon_2 v$ or $\mu_{i*} = \varepsilon_2 u + \varepsilon_1 v$ where always

$$u = \sqrt{-q + \sqrt{q^2 - \eta^3}}\,, \qquad v = \sqrt{-q - \sqrt{q^2 - \eta^3}}\,, \qquad q = \eta\left(\xi + \frac{1}{c_i}\right)\,,$$

$$\varepsilon_1 = -\frac{1}{2} + i\frac{\sqrt{2}}{2}\,, \qquad \varepsilon_1 = -\frac{1}{2} - i\frac{\sqrt{2}}{2}\,;$$

finally, it is sufficient to calculate $\nu_{i*} = \mu_{i*} + 1$. However, the choice of the solutions is limited: the Cardano formulas, for example, imply that the sum of the three real solutions should be equal to -3 (that is, to the coefficient of ν^2 in the original equation), which is not possible for $\nu \geq \nu_1 > 0$. Therefore, for each $i \in \{1, 2\}$, mark only A^1_{i*} and A^2_{i*} as the two potential stationary points $[u_{i*}, \nu_{i*}]$.

Determining the character of the stationary points by considering the second (or higher) derivatives of P would be too tedious and is not necessary in this case. It is sufficient to check on the values of $P(A)$ at the points

$$A \in \{A_*, A^1_{1*}, A^2_{1*}, A^1_{2*}, A^2_{2*}, A_1, A_2, A_3, A_4\}$$

for the smallest one (which, generally, may not be unique). Such a point $A = [c, \nu]$ then represents the most saving mixture design defined by the three values

$$(c, s, w) = (c, 1 - c(\nu + 1), c\nu)\,.$$

2.9 Free oscillation of a pole

Using Kirchhoff's theory, estimate the maximum bending of the top of an ideally fixed steel pole with a height of l, a constant section with an area of A if the pole is oscillating freely and laterally due a static lateral load P at the pole top suddenly disappearing. The positive modulus of elasticity E, the density ρ, and the inertia moment J of the material are known. Use the finite-element-method with an equidistant partition of the height l; perform the calculation for a four-step partition.

Solution: Denote by $w(x,t)$ the bending of the pole for $x \in \langle 0, l \rangle$ and $t \geq 0$, with $w(0,t) = 0$ and $w'(0,t) = 0$ at the fixed point, By Kirchhoff's theory, the equilibrium condition can be written as

$$\rho A \ddot{w} + EJw'''' = 0\,;$$

here, primes denote differentiation by x, dots by t. Nevertheless, to determine the initial bending of the pole, it is sufficient to consider w being only dependent on x using the equation

$$M = -EJw''\,,$$

as this is a statically determinate beam with a known bending moment

$$M = P(x - l)\,.$$

Particularly for $P = 0$, $w_* = 0$ everywhere with no resulting oscillation. Without loss of generality, we can assume $P > 0$; the reason is that, for $P < 0$, we would obtain the same w as for $-P$ only with an opposite sign.

Denote

$$\alpha = \sqrt{\frac{EJ}{\rho A}}\,, \qquad \beta = \frac{P}{6EJ}\,.$$

Directly integrating the equation

$$w'' = -6\beta(x - l),$$

we obtain

$$w' = -3\beta(x-l)^2 + C_1\,, \qquad w = -\beta(x-l)^3 + C_1(x-l) + C_0\,,$$

where the constants C_1 and C_0 can be calculated from the initial conditions at the fixed point $w'(0) = 0$ and $w(0) = 0$ with the result

$$C_1 = 3\beta l^2\,, \qquad C_0 = 2\beta l^3\,.$$

Thus we have

$$w = \beta\left(2l^3 - 3l^2(l-x) + (l-x)^3\right) = \beta x^2(3l - x)\,;$$

this static deviation (unlike the generally dynamic deviation w) will further be denoted by w_0. Then, clearly, $w_0(x) = w(x, 0)$ for any point x and the original equilibrium condition can be written as

$$\ddot{w} = -\alpha^2 w''''\,.$$

We will search for a solution to this partial differential equation in the form

$$w(x,t) = \sum_{i=1}^{n} \varphi_i(x)\psi_i(t)$$

for a sufficiently large n (theoretically $n \to \infty$) and a suitable system of testing functions φ_i in $\langle 0, l\rangle$ with the property $\varphi_i(0) = 0$ a $\varphi_i'(0) = 0$; the functions ψ_i in $\langle 0, \infty)$ will be calculated from a certain system of ordinary differential equations. Denoting

$$(v, u) = \int_0^l v(x)u(x)\,\mathrm{d}x\,, \qquad [v, u] = v(l)u(l) - v(0)u(0)\,,$$

with $v \in \{\varphi_1, \ldots, \varphi_n\}$ and u being a function of the variable x, which can also depend on time, integrating by parts, we obtain

$$(v, \ddot{w}) = -\alpha^2(v, w'''') = \alpha^2(v', w''') - \alpha^2[v, w''']$$
$$= -\alpha^2(v'', w'') - \alpha^2[v, w'''] + \alpha^2[v', w''] \,.$$

Since, however, $v(0) = 0$, $v'(0) = 0$, $w''(0)$ is proportionate to the bending moment at the free end of the unloaded pole and $w'''(0)$ is proportionate the the translating force at the same point, so that $w''(0) = 0$ and $w'''(0) = 0$, we have only

$$(v, \ddot{w}) = -\alpha^2(v'', w'') \,.$$

Selecting now $v = \varphi_j$ for $j \in \{1, \ldots, n\}$, respectively, we get

$$\sum_{i=1}^{n}(\varphi_j, \varphi_i)\ddot{\psi}_i = \alpha^2 \sum_{i=1}^{n}(\varphi_j'', \varphi_i'')\psi_i \,,$$

which can be written in a matrix form

$$M\ddot{\psi} = -\alpha^2 K\psi \,,$$

where the mass matrix M consists of the entries (φ_j, φ_i) and a rigidity matrix K consists of the entries (φ_j, φ_i); we aim to determine the column vector ψ, which is formed by the entries ψ_i.

For an even n, partition now the the interval $\langle 0, l\rangle$ into $n/2$ subintervals $\langle (j-1)h, jh\rangle$ where $h = 2l/n$ and $j \in \{1, \ldots, n\}$, respectively. Denote $\xi = (x - (j-1)h)/h$ and consider a function

$$\tilde{\varphi}_L(\xi) = 1 - 3\xi^2 + 2\xi^3 \,, \qquad \tilde{\varphi}_R(\xi) = 3\xi^2 - 2\xi^3 \,,$$
$$\hat{\varphi}_L(\xi) = \xi - 2\xi^2 + \xi^3 \,, \qquad \hat{\varphi}_R(\xi) = -\xi^2 + \xi^3 \,,$$

with $L = j-1$ and $R = j$. These functions, which can easily be derived using the Newton form of a Hermit cubic interpolation polynomial in $\langle (j-1)h, jh\rangle$, have the properties $\tilde{\varphi}_L(0) = \tilde{\varphi}_R(1) = h\hat{\varphi}_L(0) = h\hat{\varphi}_L(l) = 1$ while all their remaining values for $\xi \in \{0, 1\}$ are zero. If we further restricted ourselves to the interval $\langle (j-1)h, jh\rangle$ rather than $\langle 0, l\rangle$, we would, by merely integrating polynomials, obtain (even though after some effort)

$$M = \frac{h}{420}\begin{bmatrix} M_1 & M_0 \\ M_0^T & M_2 \end{bmatrix}, \qquad K = \frac{2}{h^3}\begin{bmatrix} K_1 & K_0 \\ K_0^T & K_2 \end{bmatrix},$$

where

$$M_1 = \begin{bmatrix} 156 & 22 \\ 22 & 4 \end{bmatrix}, \qquad M_2 = \begin{bmatrix} 156 & -22 \\ -22 & 4 \end{bmatrix}, \qquad M_0 = \begin{bmatrix} 54 & -13 \\ 13 & -3 \end{bmatrix},$$

$$K_1 = \begin{bmatrix} 6 & 3 \\ 3 & 2 \end{bmatrix}, \qquad K_2 = \begin{bmatrix} 6 & -3 \\ -3 & 2 \end{bmatrix}, \qquad K_0 = \begin{bmatrix} -6 & 3 \\ -3 & 1 \end{bmatrix}$$

(the non-diagonal entries in the sums $M_1 + M_2$ and $K_1 + K_2$ will disappear).

For $x \in \langle (j-1)h, jh \rangle$, we clearly have for even $j \in \{2, \dots, n\}$

$$\varphi_{j-1}(\xi) = \tilde{\varphi}_R(\xi), \qquad \varphi_j(\xi) = h\hat{\varphi}_R(\xi)$$

and similarly, for $x \in \langle jh, (j+1)h \rangle$ if $j < n$

$$\varphi_{j-1}(\xi - 1) = \tilde{\varphi}_L(\xi - 1), \qquad \varphi_j(\xi - 1) = h\hat{\varphi}_L(\xi - 1);$$

in both cases then

$$w_0(\xi) = h^3(\xi - s + 1)^2 \left(\frac{3l}{h} - \xi + s - 1 \right) = h^3(\xi - s + 1)^2 \left(\frac{n}{2} - \xi + s - 1 \right).$$

Generalizing this procedure, we obtain the band matrices

$$M = \frac{h}{420} \begin{bmatrix} M_1 & M_0 & & & \\ M_0^T & M_1 + M_2 & & & \\ & & \cdots & & \\ & & & M_1 + M_2 & M_0 \\ & & & M_0^T & M_2 \end{bmatrix},$$

$$K = \frac{2}{h^3} \begin{bmatrix} K_1 & K_0 & & & \\ K_0^T & K_1 + K_2 & & & \\ & & \cdots & & \\ & & & K_1 + K_2 & K_0 \\ & & & K_0^T & K_2 \end{bmatrix}$$

for the vector φ with $2n$ entries.

We will now try to find the vector ψ in the form

$$\psi(t) = VE_1(t)a + VE_2(t)b,$$

where a and b are column vectors of yet unknown constants, Λt is a diagonal square matrix and V is a square n by n matrix, $E_1(t)$ and $E_2(t)$ are order n matrices consisting of the entries $\cos(\Lambda_{ii}t)$ and $\sin(\Lambda_{ii}t)$, respectively, for $i \in \{1, \dots, n\}$ with

$$MV\Lambda^2 = \alpha^2 KV,$$

that is, V is the matrix of the eigenvectors of the matrix $M^{-1}K$ and Λ^2 is the matrix of the eigenvalues of the same matrix. This is because

$$\dot{\psi}(t) = V\Lambda E_1(t)a + V\Lambda E_2(t)b, \qquad \ddot{\psi}(t) = -V\Lambda^2 E_1(t)a - V\Lambda^2 E_2(t)b,$$

so that

$$MV\Lambda^2 E_1(t)a + MV\Lambda^2 E_2(t)b = KVE_1(t)a + KVE_2(t)b.$$

Moreover, by the nature of the problem, we can expect $\Lambda_{ii}^2 > 0$ for all $i \in \{1, \dots, n\}$, and so $\Lambda_{ii} > 0$ can also be chosen, which means that complex numbers do not have be considered; however, a more thorough qualitative analysis would require more space not available here. This should also be noted to the efficiency of an algorithm used to determine V and Λ^2: even if M and K are band matrices, generally, V is a full matrix and, moreover, the inverse to V is needed, too; with a small n, however, we could succesfully use a repeated power method. In this example, (see further) the `eig` MATLAB function was used to numerically calculate V and Λ^2.

Next denote by f the column vector of the entries (φ_j, w_0) for $j \in \{1, \dots, n\}$. As $E_1(0)$ and $E_2(0)$ are zero matrices, we have

$$\psi(0) = V(a+b)\,, \qquad \dot{\psi}(0) = V(b-a)\,;$$

but, at the same time, $M\psi(0) = f$ and $\dot{\psi}(0)$ must be a zero vector so that we get

$$a = b = \frac{1}{2}M^{-1}V^{-1}f\,.$$

Denoting $E = (E_1 + E_2)/2$ yields

$$\psi(t) = VE(t)V^{-1}M^{-1}f$$

and, finaly,

$$w(x,t) = \varphi^T(x)VE(t)V^{-1}M^{-1}f\,.$$

For the numerical calculation, denote further $\bar{M} = 420M/h$ and $\bar{K} = Kh^3/2$. Then $V\Lambda^2 = 210h^2\bar{M}^{-1}\bar{K}V$. The matrix V of the eigenvectors can be determined for the matrix $\bar{M}^{-1}\bar{K}$ seemingly independently of h (actually $n = l/(2h)$), the matrix Λ of the eigenvalues obtained in the same way must be multiplied by $\sqrt{210}h$. For $n = 4$, we have

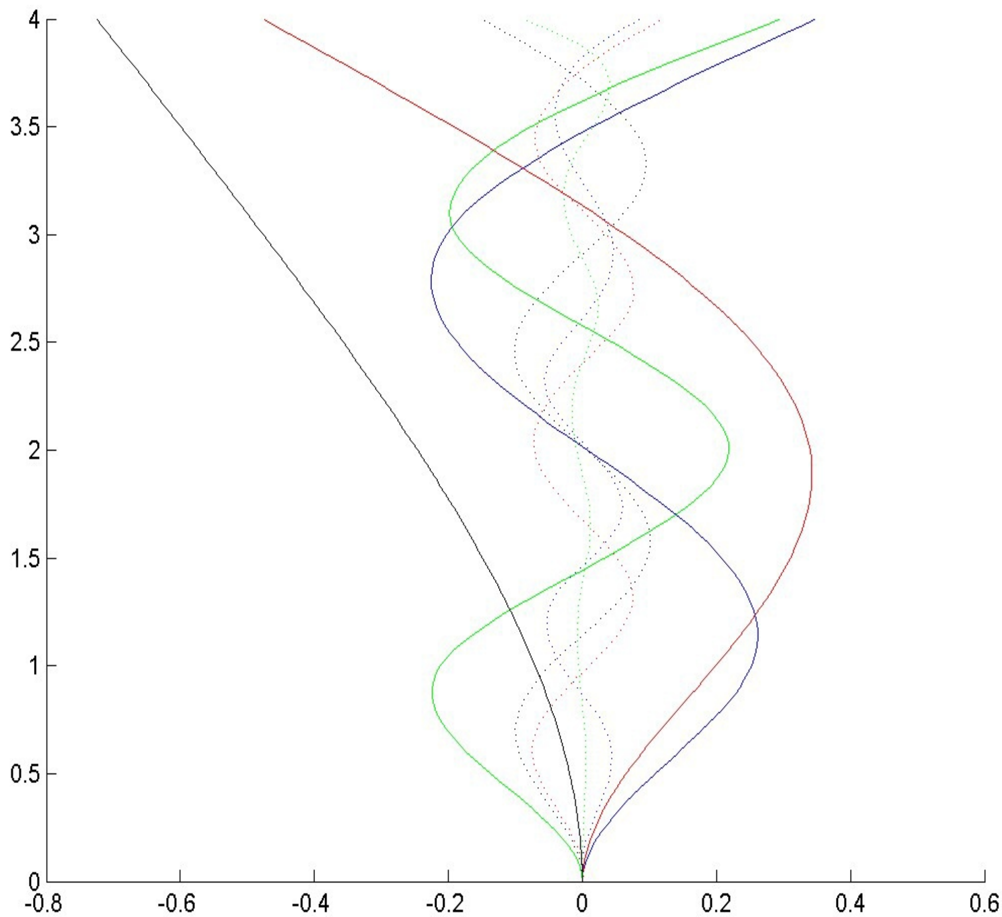
$$\bar{M} = \begin{bmatrix} 312 & 0 & 54 & -13 & & & & \\ 0 & 8 & 13 & -3 & & & & \\ 54 & 13 & 312 & 0 & 54 & -13 & & \\ -13 & -3 & 0 & 8 & 13 & -3 & & \\ & & 54 & 13 & 312 & 0 & 54 & -13 \\ & & -13 & -3 & 0 & 8 & 13 & -3 \\ & & & & 54 & 13 & 156 & 22 \\ & & & & -13 & -3 & 22 & 4 \end{bmatrix},$$

$$\bar{K} = \begin{bmatrix} 12 & 0 & -6 & 3 & & & & \\ 0 & 4 & -3 & 1 & & & & \\ -6 & -3 & 12 & 0 & -6 & 3 & & \\ 3 & 1 & 0 & 4 & -3 & 1 & & \\ & & -6 & -3 & 12 & 0 & -6 & 3 \\ & & 3 & 1 & 0 & 4 & -3 & 1 \\ & & & & -6 & -3 & 6 & 3 \\ & & & & 3 & 1 & 3 & 2 \end{bmatrix}.$$

Integrating the polynomials analytically (simple but tedious, which can be improved using, for example, MAPLE or the MATLAB `symbolic` toolbox, yields

$$\frac{210}{\beta h^2} f = [2562\ 290\ 8968\ 500\ 17094\ 626\ 11935, 5\ –1906]^{\mathrm{T}} .$$

The matrix V containing columns of the pole eigenshapes can be determined numerically: The odd rows contain the respective deviations at the points $x \in \{h, 2h, 3h, l\}$, the even rows then the respective rotations; here, the MATLAB `eig` function was used. The figure shows the eigenshapes found, corresponding to the eigenvalues sorted in descending order: first the continuous black, red, blue, and green lines are taken, then the dotted black, red, blue, and green lines. We do not include here the numerical results for Λ, V, V^{-1} a $V^{-1}M^{-1}f$ because they are too many and unclear if published in the text form; the way thy are obtained is the same. Now the entire problem has been reduced to substituting the input values into a simple algebraic formula.



The structure's eigenshapes.

The problem solution is clearly periodic, which does not correspond with the actual situation as observed: once devoid of the load, the pole eventually returns to the original position. The reason for this is that the pole has been idealized as a closed system not communicating with its environment while the energy (except for the potential and kinetic energy included in the model) actually dissipates into the surrounding environment particularly that transformed into heat through friction with the air. Under the simplest assumption of a proportional damping, the term $M\ddot{\psi}$ can be replaced by the term $M\ddot{\psi} + \varsigma M\dot{\psi}$ where ς is a positive constant, which will force a gradual dampening down of the movement; more realistic models use more complex, mostly non-linear terms.

2.10 Transition curve for a road

Design a transition curve for a road route passing from a straight line to a circular arc knowing the distance from the straight line to the circular arc h and the arc radius r. Use a cubic parabola for the transition curve. Ignore any changes in the elevation along the route.

Solution: Denote by γ the transition curve to be found, by $\tilde{k}$ the circle, and by p the straight line given. Choose Cartesian coordinates such that the point $S = [0, r]$ is the centre of k, the point $A = [-l, -h]$ lies at the intersection of p and γ, and the point $B = [b, r - \sqrt{r^2 - b^2}]$ lies at the intersection of $\tilde{k}$ and γ; clearly, the point B is on the semicircle $k \subset \tilde{k}$ such that $y \leq r$. We need to determine the remaining coordinates with unknown parameters b and l and an equation of γ containing another unknown parameter a.

First set up equations of the lines p, k, and γ in the form $y = \varphi(x)$ where φ are certain real functions with first and second derivatives needed to calculate the curvatures

$$\kappa = \frac{y''}{\sqrt{(1 + y'^2)^3}} .$$

Thus, for p, we have

$$y = -h , \qquad y' = 0 , \qquad y'' = 0 , \qquad \kappa = 0 ,$$

for k,

$$y = r - \sqrt{r^2 - x^2} , \qquad y' = \frac{x}{\sqrt{r^2 - x^2}} , \qquad y'' = \frac{r^2}{\sqrt{\left(r^2 - x^2\right)^3}} , \qquad \kappa = \frac{1}{r}$$

and, for γ,

$$y + h = a(x + l)^3 , \qquad y' = 3a(x + l)^2 , \qquad y'' = 6a(x + l) ,$$

$$\kappa = \frac{6a(x+l)}{\sqrt{(1+9a^2(x+l)^4)^3}} .$$

As, at the point A, both for p i γ, we always have

$$y(-l) = -h\,, \qquad y'(-l) = 0\,, \qquad y'(-l) = 0\,, \qquad \kappa(-l) = 0\,,$$

all we need to do is ensure the continuity of y, y', and κ at B by suitably setting the parameters a, b, and l.

The conditions of continuity for y, y', and κ at B, based on the equations γ and k are

$$a(b+l)^3 = h + r - \sqrt{r^2 - b^2}\,, \qquad 3a(b+l)^2 = \frac{b}{\sqrt{r^2-b^2}}\,,$$

$$\frac{6a(b+l)}{\sqrt{(1+9a^2(b+l)^4)^3}} = \frac{1}{r} .$$

respectively. Using the first condition, we can write

$$a = \frac{h + r - \sqrt{r^2-b^2}}{(b+l)^3}\,,$$

and, by comparing the first and second conditions, also

$$l = \frac{3\sqrt{r^2-b^2}\left(h + b - \sqrt{r^2-b^2}\right)}{b} - b\,.$$

Thus, theoretically, we can already express l and a in terms of b. Subsequently, we have to calculate b using the third condition, which can be written as

$$36a^2(b+l)^2r^2 = \left(1 + 9a^2(b+l)^4\right)^3 .$$

In this way, we obtain a single equation for the last unknown parameter b, which is, however, rather complicated so that we cannot find its analytical solution. Moreover, by a more detailed analysis, we would find that a solution might not be unique or exist in the real domain.

If a reasonable estimate b exists, however, we can try to find a numerical approximation to our engineering problem using Newton's method of tangent lines. We solve a formal equation $f(b) = 0$ where

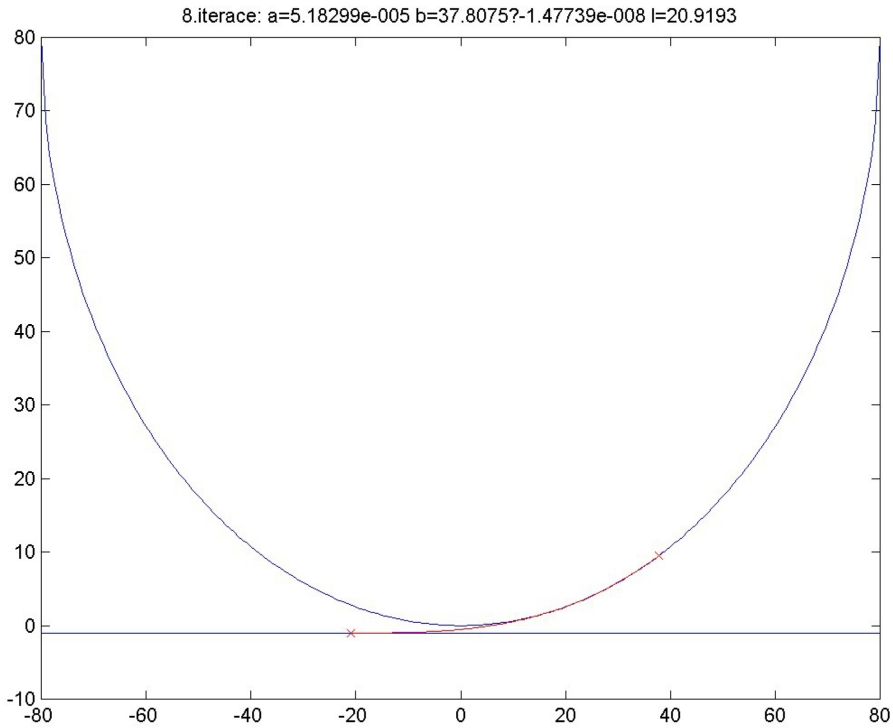
$$f(b) = 36a^2(b+l)^2r^2 - \left(1 + 9a^2(b+l)^4\right)^3$$

and l and a are already known (rather complex) functions of b. The algorithm based on the iteration scheme

$$b \leftarrow b - f(b)/f'(b)\,,$$

is clear from the below MATLAB code:

```
% cpar (C) J.V.09
% run example: rn=80;hn=1;cpar
%
syms r h b
l=3*sqrt(r^2-b^2)*(h+r-sqrt(r^2-b^2))/b-b;
a=(h+r-sqrt(r^2-b^2))/(b+l)^3;
f=36*a^2*(b+l)^2*r^2-(1+9*a^2*(b+l)^4)^3;
df=diff(f,b);
%
fil=fopen('cparf.m','w');
fprintf(fil,'fe=%s;\ndfe=%s;\n;',char(f),char(df));
fclose(fil);
%
if exist('rn','var'), r=rn; else r=80; end
if exist('hn','var'), h=hn; else h=1; end
b=r*cos(pi/4); it=0; err=Inf;
%
while abs(err)>1e-6
 if it, cparf;  bn=b-fe/dfe;
  if bn<0, bn=0; elseif bn>r, bn=r; end
  err=bn-b; b=bn; end
 l=3*sqrt(r^2-b^2)*(h+r-sqrt(r^2-b^2))/b-b;
 a=(h+r-sqrt(r^2-b^2))/(b+l)^3;
%
 xa=-r:r/100:r; ya=r-sqrt(r^2-xa.^2);
 xb=[-r,r]; yb=[-h,-h];
 xi=-l:(b+l)/100:b; yi=a*(xi+l).^3-h;
 plot(xa,ya,'b',xb,yb,'b',xi,yi,'r',...
      b,r-sqrt(r^2-b^2),'rx',-l,-h,'rx');
 buf=sprintf('%d.iterace: a=%g b=%g?%g l=%g',...
             it,a,b,err,l);
 title(buf); pause
 it=it+1; end
%
print cparf.jpg -djpeg90
```



Iteration step 8:

Here, the rather rough estimate $b = r/\sqrt{2}$ is taken for a basis with the relative error b between the last two iteration steps being tolerated up to 10^{-6}. The derivative of f' is determined formally using the `diff` function of the MATLAB `symbolic` toolbox. During the calculation, each iteration is animated. A sample result for $r = 80$ and $h = 1$ (after the eighth iteration step) is shown by the above figure.

2.11 Discharge through an orifice in a vertical wall

Determine the maximum Q_1 and minimum Q_2 discharge through an elliptic orifice with an area of A in a vertical wall. The orifice is completely beneath the water level with one of the semi-axes being vertical and the other horizontal. The orifice and the water are subject to identical pressures and the effect of the feeding rate can be neglected. The height of the water level above the centre of the orifice is h, the gravitational acceleration is g, the funnelling coefficient is ε, and the discharge rate is φ.

Solution: In the vertical wall, we will position Cartesian coordinates (x, y) so that the y-axis is vertical and the origin coincides with the centre of the orifice.

In this coordinate system, the elliptic orifice Ω is defined by the inequation

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 \le 1\,,$$

with a and b being the ellipse semi-axes (which of them is major and which minor is irrelevant). According to the problem definition, moreover, $A = \pi ab$ and $b \le h$.

For the discharge rate v, which is a function of y, under the conservation-of-energy law, (particularly, concerning the transformation of energy from the potential to the kinetic form), we have

$$\frac{1}{2}\left(\frac{v(y)}{\varphi}\right)^2 = g(h-y)\,,$$

which can be written as

$$v(y) = \varphi\sqrt{2g(h-y)}\,,$$

The above equation was described (for $\varphi = 1$) by J. E. Toricelli as early as 17th century. The total discharge through the orifice is then

$$Q = \int\int_\Omega v(y)\,\mathrm{d}x\,\mathrm{d}y\,,$$

and, after denoting $c = \varepsilon\varphi\sqrt{2g}$,

$$Q = \frac{2ca}{b}\int_{-b}^{b}\sqrt{h-y}\sqrt{b^2-y^2}\,\mathrm{d}y = 2ca\sqrt{h}\int_{-b}^{b}\sqrt{1-\frac{y}{h}}\sqrt{1-\frac{y^2}{b^2}}\,\mathrm{d}y\,.$$

Denote, finally, $\xi = b/h$; if the entire orifice is to be below the water level, then $\xi \in \langle 0, 1\rangle$. Using the substitution $y = b\cos\psi$, we obtain

$$\begin{aligned} Q &= 2cab\sqrt{h}\int_0^\pi \sqrt{1-\xi\cos\psi}\,\sin^2\psi\,\mathrm{d}\psi \\ &= \frac{2cA\sqrt{h}}{\pi}\int_0^\pi \sqrt{1-\xi\cos\psi}\,\sin^2\psi\,\mathrm{d}\psi\,. \end{aligned}$$

Generally, this integral cannot be expressed in an analytic form; even in the special case of a circle $a = b$ (or $A = \pi b^2$), the solution involves elliptic integrals. Nevertheless, one can see that it is decreasing using numeric integration such as that implemented by a simple MATLAB cycle:

```
syms psi; n=10;
for k=0:n, xi=k/n; k1=k+1;
 g=sqrt(1-xi*cos(psi))*sin(psi)^2;
 I=double(int(g,psi,0,pi));
 fprintf(1,'\nxi=%d I=%d g=%s',xi,I,char(g));
 xa(k1)=xi; ya(k1)=I; end
plot(xa,ya);
```

Here, the discharge Q only differs from the function g by being multiplied by a positive constant $2cA\sqrt{h}/\pi$.

The fact that Q is a decreasing function of ξ can also be proved analytically without numeric integration. The function

$$g(\xi) = \int_0^\pi \sqrt{1-\xi\cos\psi}\,\sin^2\psi\,\mathrm{d}\psi$$

has a derivative by ξ

$$g'(\xi) = \frac{1}{2}\int_0^\pi \frac{\cos\psi\sin^2\psi}{\sqrt{1-\xi\cos\psi}}\,\mathrm{d}\psi = -\frac{1}{2}\int_0^\pi \frac{\sin\psi\sin(2\psi)}{\sqrt{1-\xi\cos\psi}}\,\mathrm{d}\psi\,.$$

The resulting integral can formally be split into two integrals: the first one taken from 0 to $\pi/2$ while the second one from $\pi/2$ to π, with the second one, subsequently, solved using the substitution $\varsigma = \pi - \psi$. This results in

$$g'(\xi) = -\frac{1}{2}\int_0^{\pi/2} \frac{\sin\psi\sin(2\psi)}{\sqrt{1-\xi\cos\psi}}\,\mathrm{d}\psi + \frac{1}{2}\int_0^{\pi/2} \frac{\sin\varsigma\sin(2\varsigma)}{\sqrt{1+\xi\cos\varsigma}}\,\mathrm{d}\varsigma\,.$$

Adding these two integrals again yields

$$g'(\xi) = -\frac{1}{2}\int_0^{\pi/2} \sin\psi\sin(2\psi)\left(\frac{1}{\sqrt{1-\xi\cos\psi}} - \frac{1}{\sqrt{1+\xi\cos\psi}}\right)\mathrm{d}\psi\,.$$

However, since all the three terms to be integrated are positive within the integration domain for any $\xi \in \langle 0,1\rangle$, $g'(\xi) < 0$ is implied, which means that g is decreasing.

Clearly, the minimum discharge Q corresponds to the largest possible $\xi = 1$. Thus, we obtain

$$Q_1 = \frac{2cA\sqrt{h}}{\pi}\int_0^\pi \sqrt{1-\cos\psi}\,\sin^2\psi\,\mathrm{d}\psi\,,$$

by further substitution $\psi = 2\omega$, then,

$$\begin{aligned} Q_1 &= \frac{4cA\sqrt{2h}}{\pi}\int_0^{\pi/2} \sin\omega\sin^2(2\omega)\,\mathrm{d}\omega \\ &= \frac{16cA\sqrt{2h}}{\pi}\int_0^{\pi/2} \sin^3\omega\cos^2\omega\,\mathrm{d}\omega \end{aligned}$$

and finally, substituting $\cos\omega = t$,

$$\begin{aligned} Q_1 &= \frac{16cA\sqrt{2h}}{\pi}\int_0^1 \left(1-t^2\right)t^2\,\mathrm{d}t = \frac{16cA\sqrt{2h}}{\pi}\left[\frac{t^3}{3} - \frac{t^5}{5}\right]_0^1 = \frac{32cA\sqrt{2h}}{15\pi} \\ &\approx 0{,}96033740390091\,cA\sqrt{h}\,. \end{aligned}$$

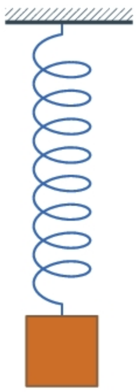
Clearly, the maximum discharge Q corresponds to the smallest possible $\xi = 0$. Although this cannot be implemented technically in a precise way, the result

$$Q_2 = \frac{2cA\sqrt{h}}{\pi}\int_0^\pi \sin^2\psi \, \mathrm{d}\psi = \frac{cA\sqrt{h}}{\pi}\int_0^\pi (1-\cos(2\psi))\, \mathrm{d}\psi$$
$$= \frac{cA\sqrt{h}}{\pi}\left[\psi + \frac{1}{2}\sin(2\psi)\right]_0^\pi = cA\sqrt{h}$$

is the well-known hydraulic formula for the discharge through a small orifice if $h \gg b$.

2.12 Harmonic oscillator

Consider an oscillator with a mass of m and a stiffness of k. (We can, for example, picture a body with a mass of m suspended on a spring with a stiffness of k oscillating about its ballanced position.)



http : //en.wikipedia.org/wiki/Harmonic_oscillator

Use the following denotations

- $x(t)$ – *the deviation* of the oscillating motion from the balanced position depending on time $t \geq 0$,
- A – *maximum deviation* of the oscillator ($|x(t)| \leq A$ at each moment t),
- T – *oscillation period* : the smallest period of time in which the deviation $x(t)$ of the periodic oscillating movement assumes two identical values,
- $f = 1/T$ – *frequency*, that is, the number of oscillations per second,
- $\omega = 2\pi f = 2\pi/T$ – *radian frequency*,
- $\varphi(t) = \omega t + \varphi_0$ – *fázi* ($\varphi(0) = \varphi_0$ is the *initial phase* at time $t = 0$).

2.12.1 Free harmonics

Based on experiments, the deviation of free harmonics at time t may be written as

$$x(t) = A \cdot \sin \varphi(t) = A \cdot \sin(\omega t + \varphi_0) \tag{2.3}$$

with a *restoring force* being at work. At time t, the restoring force $F_d = -kx$ is always directed towards the ballanced position and, by the Newton motion equation, $F_d = m \cdot a$ with an acceleration of $a = d^2x/dt^2$. Therefore

$$m\frac{d^2x}{dx^2} + kx = 0 \qquad (t \geq 0)$$

is the differential *equation of the motion of a free oscillator.* This equation can be written as

$$\frac{d^2x}{dx^2} + K^2x = 0 \qquad \left(t \geq 0,\, K^2 = \frac{k}{m} > 0\right). \tag{2.4}$$

For a solution of (2.4) in the form $x = x(t) = e^{\lambda t}$ ($\lambda \in \mathbb{C}$), the characteristic equation

$$\lambda^2 + K^2 = 0$$

has two conjugate complex roots $\lambda_{1,2} = \pm Ki$ and the functions $\sin Kt, \cos Kt$ form a fundamental system of the solutions of (2.4). Therefore, its general solution is

$$x(t) = c_1 \sin Kt + c_2 \cos Kt = C \sin(Kt + \psi), \quad c_1, c_2, C, \psi \in \mathbb{R}. \tag{2.5}$$

The following is true for the deviation (2.3) of the harmonics

$$x(t) = A \cdot \sin \varphi(t) = A \cdot \sin(\omega t + \varphi_0) = C \sin(Kt + \psi),$$

therefore,

$$C = A,\, K = \omega = \sqrt{\frac{k}{m}},\, \psi = \varphi_0 \tag{2.6}$$

and the solution of the differential equation (2.4) expresses the harmonics.

Note 2.12.1 Using the second of the equations (2.6), we can determine the frequency of the oscillation knowing the oscillator's rigidity and mass.

Note 2.12.2

- The *oscillator speed* at time t is

$$v(t) = dx(t)/dt = A\omega \cos(\omega t + \varphi_0).$$

Therefore, the oscillator will reach the maximum speed $v_{max} = |dx(t)/dt|_{max} = A\omega$ at

$$|\cos(\omega t + \varphi_0)| = 1 \Leftrightarrow \omega t + \varphi_0 = l\pi \ (l \in \mathbb{N}),$$

that is, at ballanced positions.

- The *oscillator acceleration* at time t is

$$a(t) = dv(t)/dt = -A\omega^2 \sin(\omega t + \varphi_0) = -\omega^2 x(t).$$

Therefore, the maximum acceleration $a_{max} = |dv(t)/dt|_{max} = A\omega^2$ is reached at

$$|\sin(\omega t + \varphi_0)| = 1 \Leftrightarrow \omega t + \varphi_0 = l\pi/2 \ (l \in \mathbb{N}),$$

that is, at the moment of the maximum deviation.

Note 2.12.3 Also time-independent *effective values* are used to describe an oscillator such as its deviation, speed, and acceleration.

For a function $y = f(x)$ square integrable over the interval $\langle t_1, t_2 \rangle$ the effective value is defined by

$$y_{ef} = \sqrt{\frac{1}{t_2 - t_1} \int_{t_1}^{t_2} f^2(t)\, dt}\ .$$

In the interval $\langle 0, T \rangle$ of the period, we can then determine

$$x_{ef}^2 = \frac{1}{T} \int_0^T A^2 \sin^2(\omega t + \varphi_0)\, dt = \frac{A^2}{T} \int_0^T \sin^2(\frac{2\pi}{T} t + \varphi_0)\, dt =$$

$$= \left| \begin{array}{c} \frac{2\pi}{T} t + \varphi_0 = \tau \\ \frac{2\pi}{T} dt = d\tau \end{array} \quad \begin{array}{c||c|c} t & 0 & T \\ \hline \tau & \varphi_0 & \varphi_0 + 2\pi \end{array} \right| = \frac{A^2}{2\pi} \int_{\varphi_0}^{\varphi_0 + 2\pi} \sin^2 \tau\, d\tau =$$

$$= \frac{A^2}{4\pi} \int_{\varphi_0}^{\varphi_0 + 2\pi} (1 - \cos 2\tau)\, d\tau = \frac{A^2}{8\pi} \left[2\tau - \sin 2\tau\right]_{\varphi_0}^{\varphi_0 + 2\pi} = \frac{A^2}{2},$$

$$x_{ef} = \frac{A}{\sqrt{2}}\ .$$

In much the same way

$$v_{ef}^2 = \frac{1}{T} \int_0^T A^2 \omega^2 \cos^2(\omega t + \varphi_0)\, dt = \omega^2 \cdot \left(\frac{1}{T} \int_0^T A^2 (1 - \sin^2(\omega t + \varphi_0))\, dt \right) =$$

$$= \omega^2 \cdot \left(\frac{A^2}{T} \int_0^T dt - x_{ef}^2 \right) = \omega^2 \cdot (A^2 - \frac{A^2}{2}) = \frac{\omega^2 A^2}{2},$$

$$v_{ef} = \frac{A\omega}{\sqrt{2}}.$$

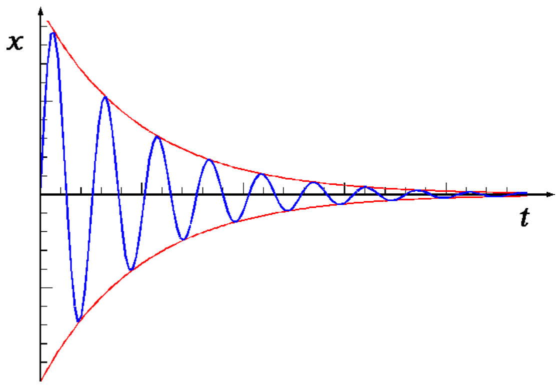
Finally,

$$a_{ef}^2 = \frac{1}{T}\int_0^T A^2\omega^4 \sin^2(\omega t + \varphi_0)\, dt = \omega^4 x_{ef}^2 = \frac{A^2\omega^4}{2},$$

$$a_{ef} = \frac{A\omega^2}{\sqrt{2}}.$$

2.12.2 Damped oscillations

Apart from a restoring force $F_d = -kx$ also a *damping force* F_t has an effect on the oscillator. The damping-force vector is usually directly proportionate to the oscillator speed vector, is oppositely oriented, which gradually dampens down the oscillation amplitude.



Thus, if we put $F_t = -R_m \frac{dx}{dt}$ where R_m is the mechanical resistance, then

$$F_d + F_t = m \cdot a$$

is true by Newton's motion equation. The *motion equation of damped oscillations* is then

$$-kx - R_m \frac{dx}{dt} = m\frac{d^2x}{dt^2}, \quad t \geq 0, \text{ tj.,}$$

$$\frac{d^2x}{dt^2} + 2\delta\frac{dx}{dt} + \omega_0^2 x = 0, \tag{2.7}$$

where $\delta = R_m/(2m)$ is a *damping coefficient* and $\omega_0 = \sqrt{k/m}$ is the *oscillator's natural frequency.*

For damped oscillations, consider the deviation

$$x(t) = A_0 f(t) \cdot \sin\varphi(t) = A_0 f(t) \cdot \sin(\omega t + \varphi_0), \tag{2.8}$$

where $\lim_{t\to\infty} f(t) = 0$ is assumed as the amplitude $A = A_0 f(t)$ decreases. Let us now solve the equation (2.7). The characteristic equation

$$\lambda^2 + 2\delta\lambda + \omega_0^2 = 0$$

has the roots

$$\lambda_{1,2} = \frac{-2\delta \pm \sqrt{4\delta^2 - 4\omega_0^2}}{2} = -\delta \pm \sqrt{\delta^2 - \omega_0^2}$$

with three different cases possible.

(ι) If $\delta < \omega_0$, the the oscillator damping is *subcritical*, $\omega_d = \sqrt{\omega_0^2 - \delta^2}$ is the *radian frequency of the sub-critical oscillations* and $\lambda_{1,2} = -\delta \pm \omega_d \cdot i$. In this case, the general solution to (2.7) will indeed be in the form (2.8),

$$x(t) = c_1 e^{-\delta t} \cos \omega_d t + c_2 e^{-\delta t} \sin \omega_d t = C e^{-\delta t} \sin(\omega_d t + \psi) \quad (c_1, c_2, C, \psi \in \mathbb{R}),$$

where $C = A_0, f(t) = e^{-\delta t}, \omega = \omega_d, \psi = \varphi_0$, that is, the function

$$x(t) = A_0 e^{-\delta t} \cdot \sin(\omega_d t + \varphi_0)$$

expresses the deviation of the damped oscillations.

($\iota\iota$) If $\delta = \omega_0$, then the oscillator damping is *critical*, $\omega_d = 0$ and $\lambda_{1,2} = -\delta < 0$ is a double real root of the characteristic equation. The general solution to (2.7) will be in the form

$$x(t) = c_1 e^{-\delta t} + c_2 t e^{-\delta t} \quad (c_1, c_2 \in \mathbb{R})$$

and cannot express the deviation of damped oscillations (2.8), in this case, there are no oscillations and, as can easily be seen, $\lim_{t\to\infty} x(t) = 0$.

($\iota\iota\iota$) Neither are there any damped oscillations if the remaining condition $\delta > \omega_0$ is true, that is, the oscillator damping is *super-critical.* The roots of the characteristic equation are real and different, the general solution to (2.7) has the form

$$x(t) = c_1 e^{(-\delta + \sqrt{\delta^2 - \omega_0^2})t} + c_2 e^{(-\delta - \sqrt{\delta^2 - \omega_0^2})t} \to 0 \text{ pro } t \to \infty \quad (c_1, c_2 \in \mathbb{R}).$$

2.12.3 Forced oscillations

Damped oscillations are effected in the sub-critical case and, their amplitude decreases with time. If oscillations are to be maintained, an external periodic *actuating force* $F_b = F_{max} \sin(\omega_b t + \varphi_{0b})$ must be added. The actuating force is repeated with a period of $T_b = 2\pi/\omega_b$ and its maximum value is F_{max}. In this way, we get an *actuated harmonic oscillator.* Oscillations produced in this

way are called *actuated* (*drived*). Like with damped oscillations, we set up the equation
$$F_d + F_t + F_b = m \cdot a.$$
The *forced oscillator motion equation* is then
$$-kx - R_m \frac{dx}{dt} + F_{max} \sin(\omega_b t + \varphi_{0b}) = m \frac{d^2x}{dt^2}, \quad t \geq 0, \text{ tj.,}$$
$$\frac{d^2x}{dt^2} + 2\delta \frac{dx}{dt} + \omega_0^2 x = \frac{F_{max}}{m} \sin(\omega_b t + \varphi_{0b}), \tag{2.9}$$
where again $\delta = R_m/(2m), \omega_0 = \sqrt{k/m}$. Mathematically, this is a linear second-order differential equation with constant coefficients and special right-hand side, which can be solved using the standard procedure.

(a) First, a general solution is found of the corresponding homogeneous equation, which corresponds to the motion equation of the damped oscillator and, therefore, has solution
$$\hat{x}(t) = A_0 e^{-\delta t} \cdot \sin(\omega_d t + \varphi_0). \tag{2.10}$$

(b) Assuming a particular solution of the differential equation (2.9) to be in the form
$$X(t) = A_b \sin(\omega_b t + \psi), \tag{2.11}$$
we receive the condition
$$A_b(\omega_0^2 - \omega_b^2) \sin(\omega_b t + \psi) + 2\delta A_b \omega_b \cos(\omega_b t + \psi) = \frac{F_{max}}{m} \sin(\omega_b t + \varphi_{0b}), \tag{2.12}$$
differentiating it by time t and simplifying
$$A_b(\omega_0^2 - \omega_b^2) \cos(\omega_b t + \psi) - 2\delta A_b \omega_b \sin(\omega_b t + \psi) = \frac{F_{max}}{m} \cos(\omega_b t + \varphi_{0b}). \tag{2.13}$$
For example, at $t = -\psi/\omega_b$, we have $\omega_b t + \psi = 0$ and equations (2.12, 2.13) are in the form
$$2\delta A_b \omega_b = \frac{F_{max}}{m} \sin(\varphi_{0b} - \psi), \ A_b(\omega_0^2 - \omega_b^2) = \frac{F_{max}}{m} \cos(\varphi_{0b} - \psi).$$
Hence
$$(2\delta A_b \omega_b)^2 + \left(A_b(\omega_0^2 - \omega_b^2)\right)^2 = \left(\frac{F_{max}}{m}\right)^2, \operatorname{tg}(\varphi_{0b} - \psi) = \frac{2\delta\omega_b}{\omega_0^2 - \omega_b^2}, \text{ tj.,}$$
$$A_b = \frac{F_{max}}{m\sqrt{(2\delta\omega_b)^2 + (\omega_0^2 - \omega_b^2)^2}}, \quad \psi = \varphi_{0b} + \operatorname{arctg} \frac{2\delta\omega_b}{\omega_0^2 - \omega_b^2}. \tag{2.14}$$

(c) The general solution to the motion equation of forced oscillations is the sum of the general solution to the corresponding homogeneous equation and a particular solution to the original non-homogeneous equation and, therefore,

$$x(t) = \hat{x}(t) + X(t) = A_0 e^{-\delta t} \cdot \sin(\omega_d t + \varphi_0) + A_b \sin(\omega_b t + \psi), \tag{2.15}$$

where A_b, ψ are determined by conditions (2.14).

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